TRANSCENDENCE OF SPECIAL VALUES OF MODULAR AND HYPERGEOMETRIC FUNCTIONS

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1. Lecture I: Modular Functions

Part 1: Most known transcendence results about functions of one and several complex variables are derived from those for commutative algebraic groups. Varieties and vector spaces are assumed defined over $\overline{\mathbb{Q}}$, although we also consider their complex points. A *commutative algebraic group*, also called a *group variety*, is a variety with a commutative group structure for which the composition and inverse maps are regular. The group $\mathbb{G}_a$ has complex points $\mathbb{C}$, under addition. The group $\mathbb{G}_m$ has complex points $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, under multiplication. A group variety G has a maximal subgroup L of the form $\mathbb{G}_a^r \times \mathbb{G}_m^s$ such that the quotient G/L is a projective group variety, otherwise known as an *abelian variety*. We will focus almost exclusively on abelian varieties and their moduli spaces. Abelian varieties of complex dimension one are called elliptic curves. The complex points $A(\mathbb{C})$ of an elliptic curve A can be represented as a complex torus $\mathbb{C}/\mathcal{L}$ where $\mathcal{L}$ is a lattice in $\mathbb{C}$. Therefore, $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for ω_1, ω_2 complex numbers with $z = \omega_2/\omega_1 \in \mathcal{H}$, where $\mathcal{H}$ is the set of complex numbers with positive imaginary part. The Weierstrass elliptic function is defined by

$$\wp(z) = \wp(z;\mathcal{L}) = \frac{1}{z^2} + \sum_{\omega\in\mathcal{L},\,\omega\neq 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Therefore, $\wp(z) = \wp(z+\omega)$, for all $\omega \in \mathcal{L}$, and $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

for certain algebraic invariants g_2, g_3 depending only on $\mathcal{L}$. The group law corresponds to addition in $\mathbb{C}/\mathcal{L}$, and is given by rational functions in the $g_2, g_3, \wp, \wp'$ with coefficients in $\mathbb{Q}$. The endomorphism algebra of $\mathbb{C}/\mathcal{L}$ is either $\mathbb{Q}$ or $\mathbb{Q}(z)$. In the latter case, z is imaginary quadratic and $\mathbb{C}/\mathcal{L}$ is said to have CM (complex multiplication).

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Part 2: We consider abelian varieties A of dimension $g \geq 1$ with a principal polarization. The set of complex points $A(\mathbb{C})$ has the structure of a complex torus $\mathbb{C}^g/\mathcal{L}$, with $\mathcal{L} = \mathbb{Z}\vec{\omega}_1 + \ldots + \mathbb{Z}\vec{\omega}_{2g}$, where $\vec{\omega}_i \in \mathbb{C}^g$ are linearly independent over $\mathbb{R}$. Moreover, we can choose the $\vec{\omega}_i$ in such a way that the *period matrices* $\Omega_1 = (\vec{\omega}_1, \ldots, \vec{\omega}_g)$, $\Omega_2 = (\vec{\omega}_{g+1}, \ldots, \vec{\omega}_{2g})$, have quotient the *normalized period matrix* $z = \Omega_2\Omega_1^{-1}$ in $\mathcal{H}_g$, the space of symmetric $g \times g$ matrices with positive definite imaginary part. The addition law on A corresponds to usual addition on $\mathbb{C}^g/\mathcal{L}$ and translates to the algebraic group law on A. The torus $\mathbb{C}^g/\mathcal{L}$ is isomorphic to $A_z(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + z\mathbb{Z}^g)$. Let

$$\mathrm{Sp}(2g, \mathbb{Z}) = \left\{\gamma \in M_{2g}(\mathbb{Z}) : \gamma \begin{pmatrix} 0_g & -I_g \\ I_g & 0_g \end{pmatrix} \gamma^t = \begin{pmatrix} 0_g & -I_g \\ I_g & 0_g \end{pmatrix} \right\},$$

where I_g and 0_g are the $g \times g$ identity and zero matrix respectively. The analytic space

$$\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathcal{H}_g$$

parameterizes the complex isomorphism classes of (principally polarized) abelian varieties of dimension g. The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$ is given by

$$z \mapsto (Az + B)(Cz + D)^{-1}, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}),$$

where A, B, C, D are in $M_g(\mathbb{Z})$. The space $\mathcal{A}_g$ is the set of complex points $V_g(\mathbb{C})$ of a quasi-projective variety V_g defined over $\overline{\mathbb{Q}}$, the Siegel modular variety. Abelian varieties in the same $\mathrm{Sp}(2g, \mathbb{Q})$-orbit are said to be isogenous. By the Poincaré irreducibility theorem, an abelian variety A is isogenous to a product of powers of simple non-isogenous abelian varieties:

$$A \widehat{=} A_1^{n_1} \times \ldots \times A_k^{n_k}, \qquad A_i \text{ simple}, \;\; A_i \not\widehat{=} A_j, \; i \neq j.$$

The endomorphism algebra $\mathrm{End}_0(A)$ of A is given by the linear maps on $\mathbb{C}^g$ preserving $\mathcal{L} \otimes \mathbb{Q}$. The endomorphism algebra is an isogeny invariant and $\mathrm{End}_0(A) = \oplus M_{n_i}(\mathrm{End}_0(A_i))$. A simple abelian variety has endomorphism algebra a division algebra over $\mathbb{Q}$ with positive involution. These have been classified and fall into the following four types. Type I: totally real number field; Type II: totally indefinite quaternion algebra over a totally real number field; Type III: totally definite quaternion algebra over a totally real number field; Type IV: central simple algebra over a CM field. A CM field is a totally imaginary quadratic extension of a totally real number field, where CM stands for complex multiplication. Notice that the endomorphism algebra of any abelian variety contains $\mathbb{Q}$, and that $\mathcal{A}_g$ is the moduli space for principally polarized abelian varieties of dimension g whose endomorphism algebra contains $\mathbb{Q}$. This is an example of a Shimura variety. When $\mathrm{End}_0(A)$ contains a CM field, we say that A has *generalized CM.* If A is simple, its endomorphism algebra will then be of Type IV. A simple abelian variety A is said to have CM when $\mathrm{End}_0(A) = K$, where K is a CM field with $[K : \mathbb{Q}] = 2\dim(A)$. An arbitrary abelian variety is said to have CM if all the simple factors in its decomposition up to isogeny have CM.

Part 3: Let $A = A_z$, $z \in \mathcal{H}_g$, be defined over $\overline{\mathbb{Q}}$. Let $H^0(A, \Omega_{\overline{\mathbb{Q}}})$ denote the holomorphic 1-forms on A defined over $\overline{\mathbb{Q}}$. Let $H^{1,0}(A) = H^0(A, \Omega_{\overline{\mathbb{Q}}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. These vector spaces have dimension g. The rank of $H_1(A, \mathbb{Z})$ is $2g$, as A is topologically the product of $2g$ circles. The periods of A are the numbers $\int_\gamma \omega$ where $\omega \in H^0(A, \Omega_{\overline{\mathbb{Q}}})$, $\gamma \in H_1(A, \mathbb{Z})$. Let $\omega_1, \omega_2, \dots, \omega_g$ be a basis of $H^0(A, \Omega_{\overline{\mathbb{Q}}})$ and $\gamma_1, \gamma_2, \dots, \gamma_{2g}$ a basis of $H_1(A, \mathbb{Z})$. For an appropriate choice of complex coordinates, we have $A(\mathbb{C}) = \mathbb{C}^g/\mathcal{L}$, where

$$\mathcal{L} = \mathbb{Z}\vec{\omega_1} + \mathbb{Z}\vec{\omega_2} + \ldots + \mathbb{Z}\vec{\omega_{2g}}, \qquad \text{for} \quad \vec{\omega_i} = \left(\int_{\gamma_i} \omega_j\right)_{j=1,\dots,g} \in \mathbb{C}^g.$$

In an analogous way, using the holomorphic differential 1-forms and the homology on a smooth projective curve X of genus g, we construct an abelian variety, called the Jacobian of X, and denoted $\mathrm{Jac}(X)$. Namely, let $K = \overline{\mathbb{Q}}(x, y)$ be an algebraic function field of one independent complex variable x, where y is related to x by an irreducible polynomial relation $P(x, y) = 0$ with coefficients in $\overline{\mathbb{Q}}$, and y actually appears. Differentiating with respect to x, we obtain $\frac{dy}{dx} = -P_x/P_y$, and see that the derivative of y with respect to x is in K. More generally, if $\varphi = Q(x, y)$ is in K, then the derivative of φ with respect to x is in K. If φ and ψ are in K, we call the expression $\varphi d\psi$ an abelian differential defined over $\overline{\mathbb{Q}}$. It can be written as $R(x, y)dx$ for some $R(x, y) \in K$. By expanding in terms of a local parameter on the Riemann surface X of y with respect to x, we can define the poles and zeros of an abelian differential, as well as their orders. An abelian differential is said to be of the first kind, or holomorphic, if it has no poles on X. We denote the differentials of the first kind defined over $\overline{\mathbb{Q}}$ by $H^0(X, \Omega_{\overline{\mathbb{Q}}})$. It is a vector space of dimension $g = \mathrm{genus}(X)$ over $\overline{\mathbb{Q}}$, and we denote $H^{1,0}(X) = H^0(X, \Omega_{\overline{\mathbb{Q}}}) \otimes \mathbb{C}$. As X is topologically the product of $2g$ circles, the module $H_1(X, \mathbb{Z})$ is of rank $2g$. Each $\gamma \in H_1(X, \mathbb{Z})$ defines an element of $H^{1,0}(X)^*$, by integration over γ. The complex torus $\mathrm{Jac}(X)$ is given by $H^{1,0}(X)^*/H_1(X, \mathbb{Z})$ and is a principally polarized abelian variety of dimension g defined over $\overline{\mathbb{Q}}$. Therefore, to each curve X there is an associated normalized period matrix $z \in \mathcal{H}_g$. Allowing complex coefficients everywhere, the Torelli locus in $\mathcal{H}_g$ is the set of z associated to the smooth projective curves X of genus g.

An automorphism α of a smooth projective curve X induces a map α^* on $H^{0,1}(X) = H^{1,0}(X)^*$ preserving $H_1(X, \mathbb{Z})$, and thus determines an element of $\mathrm{End}_0(\mathrm{Jac}(X))$. We have two representations of the endomorphism algebra of a complex abelian variety $A(\mathbb{C}) = \mathbb{C}^g/\mathcal{L}$. An endomorphism can be represented as a linear map on $\mathbb{C}^g$ preserving $\mathcal{L} \otimes \mathbb{Q}$, and therefore as an element of $M_g(\mathbb{C})$. Considering only its action on $\mathcal{L} \otimes \mathbb{Q}$, it can also be represented as an element of $M_{2g}(\mathbb{Q})$. The first representation is called the complex representation, and the second is called the rational representation. The rational representation is isomorphic to the direct sum of the complex representation with its complex conjugate.

Part 4: By way of example, we apply these considerations to the family $X_\mu(x)$, $x \in \mathbb{C}$, of smooth projective algebraic curves with affine model

$$w^p = u^{p\mu_0}(u-1)^{p\mu_1}(u-x)^{p\mu_2}$$

where μ_i, $i = 0,1,2$ are rational numbers with $0 < \mu_i < 1$ and least common denominator p, a rational prime. Let $\mu_3 = 2 - \sum_{i=0}^{2} \mu_i$. Using the Riemann–Hurwitz formula, we see that $X_\mu(x)$, $x \neq 0,1$, has genus $p-1$, whereas $X_\mu(0)$ and $X_\mu(1)$ have genus $(p-1)/2$. Let $J_\mu(x) = \mathrm{Jac}(X_\mu(x))$. There is a natural automorphism α of $X_\mu(x)$ given by $(u,w) \mapsto (u, \zeta^{-1}w)$, where $\zeta = \exp(2\pi i/p)$. This induces an endomorphism of $J_\mu(x)$ and an embedding $K = \mathbb{Q}(\zeta) \subseteq \mathrm{End}_0(J_\mu(x))$. As $[K:\mathbb{Q}] = p-1 = 2\dim(J_\mu(0))$, it follows that $H_1(J_\mu(0), \mathbb{Q})$ has no proper non-trivial K-invariant subspace. By the Poincaré irreducibility theorem, there is a simple abelian variety B such that $J_\mu(0) \widehat{=} B^s$, where $\mathrm{End}_0(B)$ is a subfield of K with $[K:F] = s$. Therefore, $J_\mu(0)$ has CM by F. For all $x \in \mathbb{C}$, there is a representation of K on $V = H^0(J_\mu(x), \Omega_{\overline{\mathbb{Q}}})$. The vector space V is a direct sum of eigenspaces V_s, $s = 1, \dots, p-1$, where the induced action of α on each $v_s \in V_s$ is multiplication by $\sigma_s(\zeta) = \zeta^s$, $\sigma_s \in \mathrm{Gal}(K/\mathbb{Q})$. The dimensions of these eigenspaces were computed by Lefschetz (see also [30]). For $x \in \mathbb{R}$, let $\langle x \rangle = x - \llcorner x \lrcorner$ denote the fractional part of x, where $\llcorner x \lrcorner$ is the largest integer less than or equal to x. For $x \neq 0,1$, we have

$$r_s = \dim(V_s) = -1 + \sum_{i=0}^{3} \langle s\mu_i \rangle.$$

Notice that $r_1 = 1$ and $r_s + r_{-s} = 2$. For $x = 0$, we have

$$r'_s = \dim(V_s) = -1 + \langle s(\mu_0 + \mu_2) \rangle + \langle s\mu_1 \rangle + \langle s\mu_2 \rangle.$$

Again, $r'_1 = 1$, but now $r'_s + r'_{-s} = 1$. The datum $\Phi = \{r_s\sigma_s\}$ is called a generalized CM type. The datum $\Phi' = \{r'_s\sigma_s\}$ is called a CM type. We assume from now on that $\mu'_0 = \mu_0 + \mu_2 < 1$, and we let $\mu'_1 = \mu_1$, $\mu'_2 = \mu_3$. The abelian varieties A of dimension $p-1$ with $K \subseteq \mathrm{End}_0(A)$ and induced representation on $H^{1,0}(A)$ isomorphic to $\Phi \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ are parameterized by $\mathcal{H}^t$, where t is the number of $s \in (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ with $r_s r_{-s} = 1$. Denoted these by $\pm s_1, \pm s_2, \dots, \pm s_t$. We obtain one such family of abelian varieties by associating to each point $(z_j)_{j=1}^t \in \mathcal{H}^t$ the lattice in $\mathbb{C}^{p-1}$ given by the vectors whose first $2t$ coordinates are $\sigma_{s_j}(u) + \sigma_{s_j}(v)z_j$, $\sigma_{-s_j}(u)z_j + \sigma_{-s_j}(v)$, $j = 1, \dots, t$, and whose last $p-1-2t$ coordinates are $\sigma_s(u)$, $\sigma_s(v)$, where s takes those values with $\dim(V_s) = 2$. Here, u and v range over $\mathbb{Z}[\zeta]$.

Example: Let $\mu_0 = \frac{2}{5}$, $\mu_1 = \frac{3}{5}$, $\mu_2 = \frac{2}{5}$, $\mu_3 = \frac{3}{5}$. Then, $p = 5$, and $r_s = 1$ for $s = 1,2,3,4$. Therefore, $t = 2$ and $\pm s_1 = 1,4$; $\pm s_2 = 2,3$. Moreover, $r'_1 = 1$, $r'_4 = 0$; $r'_3 = 1$, $r'_2 = 0$. The Jacobian $J_\mu(0)$ is a simple abelian surface with CM type $\{\sigma_1, \sigma_3\}$. For $x \neq 0,1$, the 4 dimensional $J_\mu(x)$ is isogenous to the square of an abelian surface $B(x)$ whose endomorphism algebra contains $\mathbb{Q}(\sqrt{5})$. The abelian variety $B(x)$ is said to have real multiplication, as $\mathbb{Q}(\sqrt{5})$ is a totally real field.

Part 5: Modular functions are holomorphic functions, defined on a complex symmetric domain $\mathcal{D}$, that are invariant under the action of a properly discontinuous group Γ on $\mathcal{D}$, possibly satisfying some growth conditions. We also need to normalize these functions appropriately, in order to make sense of the transcendence properties of their values at certain points. We are mainly interested in the cases where $\mathcal{D} = \mathcal{H}_g$ and $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$, for some $g \geq 1$, or $\mathcal{D} = \mathcal{H}$, and Γ is the norm unit group of a quaternion algebra. These are examples of *arithmetic groups*.

We now concentrate on these groups. We have already met $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$, and the case $g = 1$ corresponds to $\mathrm{SL}_2(\mathbb{Z})$, the two by two matrices with integer entries and determinant one. A quaternion algebra over a field F of characteristic $\neq 2$ is a central simple algebra over F of dimension 4. Each quaternion algebra is isomorphic to an algebra $A = (a, b, F)$, where $a, b \in F^*$, with basis $\{1, i, j, k\}$ satisfying,

$$i^2 = a, \quad j^2 = b, \quad k = ij = -ji.$$

The Hamiltonians $\mathbb{H}$ are the elements of $(-1, -1, \mathbb{R})$, and the matrix algebra $M_2(F)$ is $(1, 1, F)$. The quaternion algebras are non-commutative, and are division algebras if they are not isomorphic to $M_2(F)$. If $A = (a, b, F)$ is a quaternion algebra over F, and $\sigma : F \to K$ is any homomorphism from F into another field K, we define $A^\sigma = (\sigma(a), \sigma(b), \sigma(F))$. Moreover, we have $A^\sigma \otimes K = (\sigma(a), \sigma(b), K)$. Now, let F be a totally real algebraic number field of degree n over $\mathbb{Q}$. This means all n distinct embeddings σ_i, $i = 1, \ldots, n$, of F into $\mathbb{C}$ have image in $\mathbb{R}$. Suppose that σ_1 is the identity. We say that a quaternion algebra A over F is unramified at the identity, and ramified at all other infinite places, if there is an $\mathbb{R}$-isomorphism ρ_1 from $A \otimes \mathbb{R}$ to $M_2(\mathbb{R})$, whereas $A^{\sigma_i} \otimes \mathbb{R}$ is $\mathbb{R}$-isomorphic to $\mathbb{H}$, for $i = 2, \ldots, n$. For $x \in A$, let $\mathrm{Nrd}(x)$ be the determinant, and $\mathrm{Trd}(x)$ be the trace, of the matrix $\rho_1(x)$. These are called the reduced norm and reduced trace of x, respectively. An order $\mathcal{O}$ in A over F is a subring of A, containing 1, which is a free $\mathbb{Z}$-module of rank $4n$. For example, if $A = (a, b, F)$ and a, b are non-zero elements of the ring of integers $\mathcal{O}_F$ of F, then

$$\mathcal{O} = \{x = x_0 + x_1 i + x_2 j + x_3 k :, x_0, x_1, x_2, x_3 \in \mathcal{O}_F\}$$

is an order in A. In any order $\mathcal{O}$ in A, let $\mathcal{O}^1$ be the group of elements of reduced norm 1. Then, its image $\rho_1(\mathcal{O}^1)$ in $M_2(\mathbb{R})$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, and

$$\Gamma(A, \mathcal{O}) = \rho_1(\mathcal{O}^1)/\{\pm \mathrm{I}_2\}$$

is a subgroup of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{I}_2\}$. In fact, $\Gamma(A, \mathcal{O})$ is a Fuchsian group, that is, a group acting properly discontinuously on $\mathcal{H}$ with finite covolume. If Γ is a subgroup of finite index in some $\Gamma(A, \mathcal{O})$, then we call Γ a *Fuchsian group derived from a quaternion algebra A*. Two groups are *commensurable* if their intersection has finite index in each of them. If Γ is commensurable with some $\Gamma(A, \mathcal{O})$, then Γ is called an *arithmetic group.* This definition is a special case of a more general one we shall mention later.

Notes on Part 4: Let μ_i, $i=0,1,2,3$ be rational numbers satisfying

$$0<\mu_i<1, \quad i=0,1,2,3, \qquad \sum_{i=0}^{3}\mu_i=2.$$

Let $N\geq 2$ be the least common denominator of the μ_i. For $x\neq 0,1,\infty$, consider the smooth projective curve $X_\mu(x)$ with affine model

$$w^N=u^{N\mu_0}(u-1)^{N\mu_1}(u-x)^{N\mu_2}.$$

There is a natural automorphism α of $X_\mu(x)$ given by $(u,w)\mapsto(u,\zeta^{-1}w)$, where $\zeta=\exp(2\pi i/N)$. This induces a representation of $K=\mathbb{Q}(\zeta)$ on the vector space $V=H^0(J_\mu(x),\Omega_{\overline{\mathbb{Q}}})$, where $J_\mu(x)=\mathrm{Jac}(X_\mu(x))$. We consider the eigenspaces V_s, $s\in(\mathbb{Z}/N\mathbb{Z})^*$ (s is coprime to N), where the induced action of α on each $v_s\in V_s$ is multiplication by $\sigma_s(\zeta)=\zeta^s$, $\sigma_s\in\mathrm{Gal}(K/\mathbb{Q})$. Following [30], there is a basis of V_s of the form,

$$\omega=\omega(s,\alpha,\beta,\gamma)=\frac{u^\alpha(u-1)^\beta(u-x)^\gamma du}{w^s}$$

for integers α,β,γ. Using local parameters $u=z^N$ at $u=0$, $(u-1)=z^N$ at $u=1$, $(u-x)=z^N$ at $u=x$, and $u^{-1}=z^N$ at $u=\infty$, we deduce that,

$$\begin{aligned}
\mathrm{ord}_0\,\omega &= N(\alpha+1)-sN\mu_0-1\\
\mathrm{ord}_1\,\omega &= N(\beta+1)-sN\mu_1-1\\
\mathrm{ord}_x\,\omega &= N(\gamma+1)-sN\mu_2-1\\
\mathrm{ord}_\infty\,\omega &= s(N\mu_0+N\mu_1+N\mu_2)-N(\alpha+\beta+\gamma+1)-1.
\end{aligned}$$

In order for ω to be holomorphic, we need

$$\begin{aligned}
\alpha &\geq N^{-1}(sN\mu_0+1)-1\\
\beta &\geq N^{-1}(sN\mu_1+1)-1\\
\gamma &\geq N^{-1}(sN\mu_2+1)-1\\
\alpha+\beta+\gamma &\leq N^{-1}(s(N\mu_0+N\mu_1+N\mu_2)-1)-1.
\end{aligned}$$

For a real number x, let $\llcorner x\lrcorner$ be the largest integer less than or equal to x, and let $\langle x\rangle=x-\llcorner x\lrcorner$ be the fractional part of x. Therefore, $x=\llcorner x\lrcorner+\langle x\rangle$. If an integer is at least $-x=-\llcorner x\lrcorner-\langle x\rangle$, then it must be at least $-\llcorner x\lrcorner$, since $0\leq\langle x\rangle<1$. Moreover, if an integer is at most x, then it is at most $\llcorner x\lrcorner$. Therefore,

$$\begin{aligned}
\alpha &\geq -\llcorner 1-s\mu_0-\tfrac{1}{N}\lrcorner\\
\beta &\geq -\llcorner 1-s\mu_1-\tfrac{1}{N}\lrcorner\\
\gamma &\geq -\llcorner 1-s\mu_2-\tfrac{1}{N}\lrcorner\\
\alpha+\beta+\gamma &\leq \llcorner s(\mu_0+\mu_1+\mu_2)-\tfrac{1}{N}\lrcorner-1.
\end{aligned}$$

Together, these inequalities imply the following bounds for $t=\alpha+\beta+\gamma$:

$$-\llcorner 1-s\mu_0-\frac{1}{N}\lrcorner-\llcorner 1-s\mu_1-\frac{1}{N}\lrcorner-\llcorner 1-s\mu_2-\frac{1}{N}\lrcorner\leq t\leq\llcorner s(\mu_0+\mu_1+\mu_2)-\frac{1}{N}\lrcorner-1.$$

The first set of inequalities also imply (weaker) upper and lower bounds for t whose difference is $2(1-\frac{2}{N})$. As $0 \leq 2(1-\frac{2}{N}) < 2$, the integer t can take *at most* two values, its possible minimal value and its possible minimal value plus 1. For given values of (α,β,γ), the choices $(\alpha+1,\beta,\gamma)$, $(\alpha,\beta+1,\gamma)$ and $(\alpha,\beta,\gamma+1)$ lead to linearly dependent differential forms. Therefore, only the possible values of t need to be counted. The number of linearly independent differential forms ω is the number of integers satisfying the last displayed inequality for t. We deduce that $\dim(V_s)$ is either $0, 1,$ or 2, and in any case equals

$$\llcorner s(\mu_0+\mu_1+\mu_2)-\frac{1}{N}\lrcorner + \llcorner 1-s\mu_0-\frac{1}{N}\lrcorner + \llcorner 1-s\mu_1-\frac{1}{N}\lrcorner + \llcorner 1-s\mu_2-\frac{1}{N}\lrcorner$$

Rewriting this in terms of fractional parts, it is straightforward to deduce that

$$\dim(V_s) = \langle s\mu_0\rangle + \langle s\mu_1\rangle + \langle s\mu_2\rangle - \langle s\mu_0+s\mu_1+s\mu_2\rangle,$$

which, using $\mu_3 = 2-(\mu_0+\mu_1+\mu_2)$, can be written as

$$r_s = \dim(V_s) = -1 + \langle s\mu_0\rangle + \langle s\mu_1\rangle + \langle s\mu_2\rangle + \langle s\mu_3\rangle.$$

Similar considerations lead to the expression for r'_s given in Part 4.

2. Lecture II: Transcendence of Special Values of Modular Functions

Part 1: Many classical transcendence results follow from the Schneider–Lang Theorem. Let ρ be a real number. An entire complex function F is said to be of order at most ρ if there is a constant $C > 0$ such that, for all R sufficiently large,

$$|F|_R = \max_{|z|=R} |F(z)| \leq C^{R^\rho}.$$

A meromorphic function is said to be of order at most ρ if it can be expressed as a quotient of entire functions of order at most ρ.

Theorem 2.1. *Let K be a number field and $f_1, \ldots, f_N$ meromorphic functions of order at most ρ. Assume that the field $K(f_1, \ldots, f_N)$ has transcendence degree at least two over K (that is, at least two of the f_i are algebraically independent over K). Suppose that the ring $K[f_1, \ldots, f_N]$ is stable under differentiation with respect to z, that is, under $D = d/dz$. Then, there are only finitely many complex numbers w such that $f_i(w) \in K$ for all $i = 1, 2, \ldots, N$.*

Summary of the proof of Schneider–Lang: Let f, g be two functions among the f_i which are algebraically independent over K. We let F be the function $\sum_{i,j=1}^r a_{ij} f^i g^j$, where the a_{ij} are *unknowns* in the ring of integers $\mathcal{O}_K$ of K. We use a result of effective linear algebra, called *Siegel's Lemma*, to find a_{ij} with controled arithmetic size such that $D^k F(w_\ell) = 0$ for all $k < n$ and for m points w_ℓ at which the $f_i(w_\ell) \in K$. The r, m, n are to be chosen later with $r^2 = 2nm$, so that the number of unknowns a_{ij} is twice the number of equations $D^k F(w_\ell) = 0$. As f, g are algebraically independent over K, the function F is not identically zero. Thus, there is a smallest integer $s \geq n$ such that $\alpha = D^s F(w_\ell) \neq 0$ for some ℓ. The number α is a non-zero element of K whose arithmetic size we can control. This gives an "arithmetic" lower bound for $\log|\alpha| \geq -[K : \mathbb{Q}]\mathrm{size}(\alpha)$, and here it is essential that $\alpha \neq 0$. On the other hand, we have an "analytic" upper bound for $\log|\alpha|$ using the maximum–modulus principle and the fact that F has high order zeros at the w_ℓ. For appropriate choices of the r, m, n in terms of s the arithmetic and analytic bounds contradict each other.

Example: Let τ be an algebraic number with positive imaginary part and $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_2/\omega_1$. Suppose the underlying equation of $\mathbb{C}/\mathcal{L}$ is of the form $y^2 = 4x^3 - g_2 x - g_3$, with g_2, g_3 algebraic. Let K be the number field $\mathbb{Q}(\tau, g_2, g_3)$. The corresponding Weierstrass elliptic function satisfies $(x, y) = (\wp(z), \wp'(z)) \in E$, therefore $K[\wp(z), \wp'(z), \wp(\tau z), \wp'(\tau z)]$ is stable under differentiation. The functions $\wp$, $\wp'$ have order two. Moreover, the group law on E corresponds to an addition law for $\wp$ and $\wp'$ defined by rational functions with coefficients in K. We apply Theorem 2.1 to the points $m\omega_1/2$, where m is an odd integer and deduce that $\wp(z)$, $\wp(\tau z)$ are algebraically dependent. This implies that τ maps $\mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$ into itself, and therefore is quadratic imaginary over $\mathbb{Q}$. (Schneider, 1937)

<u>Part 2:</u> This Example gives a transcendence result for special values of the classical modular function. This is the function $j = j(\tau)$ on $\mathcal{H}$, invariant with respect to the action of $\mathrm{PSL}(2,\mathbb{Z})$ given by

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

In particular, $j(\tau) = j(\tau+1)$ for all $\tau \in \mathcal{H}$, and $j(\tau)$ is uniquely determined by the first two terms in its Fourier expansion:

$$j(\tau) = \exp(-2\pi i\tau) + 744 + \sum_{n=1}^{\infty} a_n \exp(2\pi i n\tau).$$

For example, $a_1 = 196884$, $a_2 = 21493760$. For all $n \geq 1$, the coefficient a_n is a positive integer. From Example 3 we have,

$$\{\tau \in \mathcal{H} : \tau \in \overline{\mathbb{Q}} \text{ and } j(\tau) \in \overline{\mathbb{Q}}\} = \{\tau \in \mathcal{H} : [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}.$$

To see this, recall that to every $\tau \in \mathcal{H}$ we associate the complex torus of dimension one:

$$A_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

This torus has the underlying structure of a projective cubic curve of the form $y^2 = 4x^3 - g_2 x - g_3$, where $g_2, g_3 \in \mathbb{Q}(j(\tau))$. Its endomorphism algebra $\mathrm{End}_0(A_\tau) = \mathrm{End}(A_\tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ consists of multiplications by the numbers α preserving $\mathbb{Q} + \tau\mathbb{Q}$. This algebra is either $\mathbb{Q}$ or $\mathbb{Q}(\tau)$. In the latter case, the number τ must be imaginary quadratic, and we say that A_τ has *complex multiplication*. Indeed, the conditions $\alpha = a + b\tau$ and $\alpha\tau = c + d\tau$, for $a, b, c, d \in \mathbb{Q}$, $\alpha \neq 0$, imply a quadratic equation over $\mathbb{Q}$ for τ:

$$\tau = \frac{\alpha\tau}{\alpha} = \frac{c + d\tau}{a + b\tau}, \qquad \text{i.e.} \quad b\tau^2 + (a-d)\tau - c = 0.$$

When $\alpha \in \mathbb{Q}$, then $b = c = 0$, $a = d$ and the quadratic equation is trivial. Otherwise, $\alpha \in \mathbb{Q}(\tau)$, with τ imaginary quadratic.

There is another way to look at the Example following Theorem 2.1. Recall that τ and $j(\tau)$ are assumed to be algebraic numbers. The linear subspace of $\mathbb{C}^2$ given by $W = \{(z_1, z_2) : z_1 - \tau z_2 = 0\}$ is defined over $\overline{\mathbb{Q}}$ and contains the point (ω_2, ω_1). This point is in the kernel of the exponential map of $E \times E$. The space W also contains the rational multiples of (ω_2, ω_1) whose images under the exponential map give rise to non-trivial algebraic points on $E \times E$. Suppose we know that $H = \exp(W)$ is not only a Lie subgroup of $E \times E$ but is also a connected *algebraic* subgroup of dimension one defined over $\overline{\mathbb{Q}}$. Then, H is isogenous to E (there is a surjection from H to E with finite kernel). In fact, as $\omega_1, \omega_2 \neq 0$, the projections p_1, p_2 from H to the factors of $E \times E$ are isogenies. Therefore $p_2 \circ p_1^{-1}$ is well-defined as an element of the endomorphism algebra $\mathrm{End}_0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ of E. Its lift to the tangent space of E is a rational multiple of τ. Therefore τ leaves the space $\mathbb{Q} + \mathbb{Q}\tau$ invariant and is therefore imaginary quadratic.

Part 3: For at least forty years many transcendence results have been viewed as criteria for an analytic subgroup of a group variety to be algebraic. Roughly speaking, we ask that the analytic subgroup contain sufficiently many algebraic points. In particular, Lang, Bombieri, Nesterenko, Masser, Brownawell, Waldschmidt and many others addressed such problems. The most far-reaching result of this type was obtained by Wüstholz in the 1980's. Let G be a connected commutative algebraic group defined over $\overline{\mathbb{Q}}$. Let $T_e(G)$ be its tangent space at the origin. This is the Lie algebra of invariant derivations, and has a natural $\overline{\mathbb{Q}}$-structure. Let A be an analytic subgroup of $G(\mathbb{C})$. Typically, this subgroup is the image of a group homomorphism $\varphi : \mathbb{C}^d \to G(\mathbb{C})$, called a d-parameter subgroup. This subgroup may not have the subspace topology inherited from $G(\mathbb{C})$ and may not be closed. We say that A is defined over $\overline{\mathbb{Q}}$ if its Lie algebra is defined over $\overline{\mathbb{Q}}$ as a vector space. The analytic subgroup theorem of Wüstholz gives a criterion for A to contain an algebraic subgroup.

Theorem 2.2. *An analytic subgroup defined over $\overline{\mathbb{Q}}$ contains a non-trivial algebraic subgroup defined over $\overline{\mathbb{Q}}$ if and only if it contains a non-trivial algebraic point.*

Where did the typical transcendence argument go? The answer is that all *current* proofs of such results are extremely deep elaborations of the proof of the Schneider–Lang theorem. For example, they require generalizing Baker's analytic methods and, for the non-vanishing step, using hard commutative algebra techniques introduced by Brownawell, Masser and Wüstholz. A rough outline of the proof is given in the notes at the end of this lecture. These theorems imply that all $\overline{\mathbb{Q}}$-linear relations between abelian periods come from isogenies. For simplicity, we only consider periods of differentials of the first kind.

Theorem 2.3. *Let A and B be abelian varieties over $\overline{\mathbb{Q}}$, and denote by V_A the $\overline{\mathbb{Q}}$-vector subspace of $\mathbb{C}$ generated by all periods $\int_\gamma \omega$ with $\gamma \in H_1(A, \mathbb{Z})$ and differentials of the first kind $\omega \in H^0(A, \Omega_{\overline{\mathbb{Q}}})$. Then, $V_A \cap V_B \neq \{0\}$ if and only if there are non-trivial simple abelian subvarieties A' of A and B' of B with A' isogenous to B', written $A' \widehat{=} B'$.*

Moreover, the period relations on a given abelian variety are all induced by endomorphisms of A.

Theorem 2.4. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$. By the Poincaré irreducibility theorem, $A \widehat{=} A_1^{k_1} \times \ldots \times A_s^{k_s}$ with simple, pairwise non-isogenous abelian varieties A_i defined over $\overline{\mathbb{Q}}$. We have,*

$$\dim_{\overline{\mathbb{Q}}}(V_A) = \sum_{i=1}^{s} \frac{2\dim_{\mathbb{C}}(A_i)^2}{\dim_{\mathbb{Q}}(\mathrm{End}_0(A_i))}.$$

Part 4: By the Poincaré irreducibility theorem, the abelian variety A_z is isogenous to a product of powers of simple non-isogenous abelian varieties:

$$A_z \widehat{=} A_1^{n_1} \times \ldots \times A_k^{n_k}, \qquad A_i \text{ simple}, \quad A_i \not\widehat{=} A_j, \; i \neq j.$$

The endomorphism algebra $\mathrm{End}_0(A_i) = \mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a simple abelian variety is a division algebra over $\mathbb{Q}$ with positive involution. There are many more possibilities for such division algebras than in the case $g = 1$. When this division algebra is a number field $\mathbb{L}$ with $[\mathbb{L} : \mathbb{Q}] = 2\dim(A_i)$, we say that A_i has CM (complex multiplication). We say that A_z has CM when all of its factors A_i have CM. The corresponding point $z \in \mathcal{H}_g$, or its $\mathrm{Sp}(2g, \mathbb{Z})$-orbit in $\mathcal{A}_g$, is said to be a CM or special point. The abelian varieties defined over $\overline{\mathbb{Q}}$ correspond to the points of $V_g(\overline{\mathbb{Q}})$, and include all the CM abelian varieties. Moreover, the CM abelian varieties have normalized period matrix in $M_g(\overline{\mathbb{Q}})$. Conversely, we have the higher dimensional analogue of Schneider's theorem, proved jointly by Shiga, Wolfart, and myself [6],[26] using modern transcendence techniques, especially results of Wüstholz [32].

Theorem 2.5. *Let $J : \mathcal{H}_g \to V_g(\mathbb{C})$ be a holomorphic $\mathrm{Sp}(2g, \mathbb{Z})$-invariant map with $J(z) \in V_g(\overline{\mathbb{Q}})$ for all CM points z. Then the exceptional set of J, given by*

$$\mathcal{E} = \{z \in \mathcal{H}_g \cap M_g(\overline{\mathbb{Q}}) : J(z) \in V_g(\overline{\mathbb{Q}})\}$$

consists exactly of the CM points.

Therefore, the special values $J(z)$, $z \in \mathcal{H}_g \cap M_g(\overline{\mathbb{Q}})$, are "transcendental", that is, not in $V_g(\overline{\mathbb{Q}})$, whenever z is not a CM point. Refinements of this result were obtained by my student G. Derome [11].

As in the case $g = 1$, the CM points are important. One reason is that the action of the absolute Galois group on the torsion points of CM abelian varieties, and on the values of $J(z)$ at CM points, is well understood. This is a basic tool for studying the arithmetic of abelian varieties and modular forms. A natural question is to ask how the image $\overline{\mathcal{E}}$ in $\mathcal{A}_g$ of the exceptional set $\mathcal{E}$ of J is distributed.

Conjecture 2.6. *Let Z be an irreducible algebraic subvariety of $\mathcal{A}_g$. Then $Z \cap \overline{\mathcal{E}}$ is a Zariski dense subset of Z iff Z is a special subvariety of $\mathcal{A}_g$.*

Shimura subvarieties are moduli spaces for polarized abelian varieties with a fixed extra structure on their endomorphism ring and torsion points. The special subvarieties are irreducible components of their images under the correspondences coming from $\mathrm{Sp}(2g, \mathbb{Q})$. André–Oort [1], [23] made the analogous celebrated conjecture for CM points almost 20 years ago. By Theorem 2.5, Conjecture 2.6 is equivalent to André–Oort, which Klingler, Ullmo and Yafaev recently announced follows from GRH.

Part 5: We apply these considerations to the symplectic modular function. First, recall Schneider's theorem on the classical j-function which says that $z, j(z) \in \overline{\mathbb{Q}}$ only at the CM points. Let $J : \mathcal{H}_g \mapsto V_g(\mathbb{C})$ be the generalized J-function of Theorem 2.5. We have $J(z) \in V_g(\overline{\mathbb{Q}})$ precisely when the abelian variety A_z is defined over $\overline{\mathbb{Q}}$. In particular, the above results apply to A_z. If we also have $z \in M_g(\overline{\mathbb{Q}})$, then there are $\overline{\mathbb{Q}}$-linear relations between the entries of the unnormalized period matrices Ω_1, Ω_2 with elements in V_A. This follows from $\Omega_2 = \Omega_1 z$. By Theorem 2.4, these relations come from endomorphisms. We then have to show that the $\overline{\mathbb{Q}}$-linear relations $\Omega_2 = \Omega_1 z$ give rise to enough endomorphisms to ensure that A_z has CM. This last step requires a close look at Shimura's construction of his varieties of PEL type [27]. The details are contained in two papers on joint work by Shiga, Wolfart and me [6],[26].

Indeed, with the notation of Lecture 1, Parts 2 and 3, recall that there is a basis ω_i, $i = 1, \ldots, g$, of $H^0(A, \Omega_{\overline{\mathbb{Q}}})$ such that

$$\Omega_1 z = \left(\int_{\gamma_j} \omega_i\right) z = \Omega_2 = \left(\int_{\gamma_{j+g}} \omega_i\right),$$

where $i = 1, \ldots, g$ indexes the rows and $j = 1, \ldots, g$ indexes the columns. Therefore, when $z \in M_g(\overline{\mathbb{Q}})$, we have linear dependence relations over $\overline{\mathbb{Q}}$ of the form,

$$z_{1j} \int_{\gamma_1} \omega_i + z_{2j} \int_{\gamma_2} \omega_i + \ldots + z_{gj} \int_{\gamma_g} \omega_i = \int_{\gamma_{j+g}} \omega_i.$$

Suppose that A_z is simple. If $\mathrm{End}_0(A_z) = \mathbb{Q}$, then by Theorem 2.3 there can be no such period relations. Therefore $\mathrm{End}_0(A_z) \supsetneq \mathbb{Q}$. For the general case, we show that $z \in M_g(\overline{\mathbb{Q}})$ in fact implies non-trivial linear dependence relations over $\overline{\mathbb{Q}}$ between the vector components of an $\mathrm{End}_0(A_z)$-basis of $\mathbb{Q}^g + z\mathbb{Q}^g$. This requires working with Shimura's more general moduli spaces for abelian varieties with extra endomorphism structure. Once again, one uses Theorem 2.3 to show that such relations can't exist unless A_z belongs to a moduli space of dimension zero, which is precisely the CM case. This is the flavor of the argument in [26].

The argument in [6] is a little different, although the basic tool is still Theorem 2.3, or rather its formulation as Theorem 2.4. The two conditions $z \in M_g(\overline{\mathbb{Q}})$ and A_z is defined over $\overline{\mathbb{Q}}$ lead to two different representations of $\mathrm{End}_0(A_z)$ in $M_g(\overline{\mathbb{Q}})$. The first representation arises from a choice of $\overline{\mathbb{Q}}$-basis of a lattice of A_z, using the fact that $z \in M_g(\overline{\mathbb{Q}})$. The second representation comes from the choice of a $\overline{\mathbb{Q}}$-basis of the tangent space at the origin of A_z, using the fact that A_z is defined over $\overline{\mathbb{Q}}$. We can reduce to the case of simple A_z. We use Shimura's paper [27] to show that these two representations are "intertwined" by a matrix with less than $2g^2/[\mathrm{End}_0(A_z) : \mathbb{Q}]$ non-zero entries, except in the CM case. Therefore, using Theorem 2.4, the abelian variety A_z must have CM.

Notes on Part 1: Several classical results are corollaries of the Schneider–Lang Theorem:

Example 1: Suppose $\alpha \neq 0$ is an algebraic number with e^{α} algebraic. Let $K = \mathbb{Q}(\alpha, e^{\alpha})$ and $f_1(z) = z$, $f_2(z) = e^{\alpha z}$. These are algebraically independent functions such that $K[f_1, f_2]$ is stable under differentiation. *Using the fact that* $e(z_1 + z_2) = e(z_1)e(z_2)$, we see that f_1 and f_2 take values in K at all points $m\alpha$ for $m \in \mathbb{Z}$. There are infinitely many such points, contradicting Theorem 2.1. Therefore, $e^{\alpha} \notin \overline{\mathbb{Q}}$, for $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$. (Hermite–Lindemann, 1873–1882))

Example 2: Suppose $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$ are algebraic with α^{β} algebraic. Let $K = \mathbb{Q}(\alpha, \beta, \alpha^{\beta})$ and $f_1(z) = e^{z}$, $f_2(z) = e^{\beta z}$. As $\beta \notin \mathbb{Q}$, these are algebraically independent functions. Moreover, they generate over K a ring stable under differentiation. Again using the addition law for the exponential function, we see that f_1 and f_2 take values in K at all points $m \log \alpha$, for any determination of $\log \alpha$, contradicting Theorem 2.1. Therefore α^{β} is transcendental. (Gelfond–Schneider, 1934)

Notes on Part 3: We give an outline of the proof of Theorem 2.2. First, we restate it as follows.

Theorem 2.7. *Let $u \in \exp_G^{-1}(G(\overline{\mathbb{Q}}))$ and Z_u be the smallest $\overline{\mathbb{Q}}$-vector space of $T_e(G)$ containing u. Then $Z_u = T_e(H_u)$ for some (unique) connected algebraic group subvariety of G itself defined over $\overline{\mathbb{Q}}$.*

Step 1. Let n be the dimension of G. We can embed G in a projective space $\mathbb{P}^N$ and work with the analytic functions on $\mathbb{C}^n$ obtained by composing this embedding with the exponential map $\exp_G : \mathbb{C}^n \simeq T_e(G)(\mathbb{C}) \to G(\mathbb{C})$. These functions have order at most 2. The corresponding affine coordinates define functions that generate a ring over $\overline{\mathbb{Q}}$ stable under differentiation by the complex coordinates of the tangent space to the origin of G.
Step 2. Assume that $\dim(Z_u) = n - 1$, the case in most applications. Given a homogeneous polynomial $P = P(X_0, \ldots, X_N)$, not vanishing on all of G, let $F(z) = P(\exp_G(z))$, for $z \in T_e(G)(\mathbb{C})$. Using Siegel's Lemma, we construct a P, with controlled coefficients, such that $F(z)$ vanishes to high order "along Z_u" at many points of $\mathbb{Z}u$. "Along Z_u" refers to the vanishing of $\partial^t F(z)$, where $\partial^t = \partial_1^{t_1} \ldots \partial_{n-1}^{t_n}$ for a basis ∂_i, $i = 1, \ldots, n-1$, of Z_u.
Step 3. Step 2 implies that $\partial^t F(su)$ is small for a yet larger range of t, s. This analytic upper bound contradicts an arithmetic lower bound if these numbers are non-zero. Therefore, $F(z)$ has zeros of even larger order "along Z_u" at even more points of $\mathbb{Z}u$. In fact, things are too good as we have a resulting over-determined linear system in the coefficients of P.
Step 4. By Step 3, the point u must have some special properties that ensure the linear system is not, in fact, contradictory for this particular u. The exact meaning of this is contained in the "multiplicity estimates". They imply that there is a connected algebraic group variety $H \neq G$ such that

the cardinality S_H of $[s]_G \exp_G(u)$ modulo H is not too large. Moreover, the order of vanishing in Step 3 is controlled by $\dim(Z_u \cap T_e(H))$.
Step 5. If $H = 0$, then S_H is large, contradicting Step 4. Therefore $H \neq 0$. If $T_e(H) \not\subseteq Z_u$, then $\dim(Z_u \cap T_e(H))$ is small enough that Step 4 leads to a bound on the order of vanishing that contradicts Step 3. Finally, an induction argument on the dimension of G shows that $T_e(H) = Z_u$.

Notice the features in common with the applications of the Schneider–Lang Theorem. We have algebraically independent analytic functions of several complex variables, with controlled growth, generating a ring closed under partial differentiation. Moreover, we generate many algebraic points from a single one using the group law. There are quantitative versions of the above results. For example, Baker's bounds for lower bounds for linear forms in logarithms can be recovered.

For want of a reference, Shiga, Wolfart and I [6], [26] published details of the proofs of Theorems 2.3, 2.4. These results were announced by Wüstholz [32] and it was general folklore that they followed from his analytic subgroup theorem. The argument resembles our alternative discussion of the Example following the Schneider-Lang Theorem. If $V_A \cap V_B \neq \{0\}$, there is a linear relation over $\overline{\mathbb{Q}}$ between periods on $G = A \times B$. This defines a subspace Z of $T_e(G)$, defined over $\overline{\mathbb{Q}}$, and containing an element u of $\exp_G^{-1}(G(\overline{\mathbb{Q}}))$. By Wüstholz's theorem, there is a proper connected algebraic subgroup H of G, defined over $\overline{\mathbb{Q}}$, with $u \in T_e(H) \subseteq Z$. We then reduce to the case where $A \widehat{=} H \subseteq A^m$, with A simple, and all the coordinates of u are non-zero. The projections p_μ of A^m to each of its factors map H onto A. The $p_\mu \circ p_\nu^{-1}$ therefore define elements of $\mathrm{End}_0(A)$ and lift to define the equations of $T_e(H)$. This shows that the period relations are induced by endomorphisms of A.

3. Lecture III: Hypergeometric functions

Part 1: Many different functions are called “hypergeometric functions”. The oldest is the classical hypergeometric function of one complex variable, which occurs in many different branches of mathematics. This function is also known as the Gauss hypergeometric function. It is a solution of a second order linear homogeneous ordinary differential equation with three regular singular points at $0, 1, \infty$. In 1857, Riemann established that a function with two linearly independent branches at the points $0, 1, \infty$, and with suitable branching behavior at these points, necessarily satisfies a hypergeometric differential equation and hence is itself a hypergeometric function. This is part of Riemann's famous “viewpoint” that characterizes analytic functions by their behaviour at singular points. In 1873, Schwarz determined the list of algebraic Gauss hypergeometric functions. He also established necessary conditions for the quotient of two solutions of a classical hypergeometric differential equation to be invertible, with inverse an automorphic function on the unit disk. In the 1880's, Picard generalized Riemann's approach to the two variable Appell hypergeometric functions. The $n > 2$ variable generalizations are known as Lauricella functions.

Let a,b,c be complex numbers with $c \neq 0$ and not a negative integer. Consider the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (c-(a+b+1)x)\frac{dy}{dx} - aby = 0. \tag{3.1}$$

Euler introduced the following series solution to this equation

$$F = F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \qquad |x| < 1, \tag{3.2}$$

where for any complex number w we define $(w,n) = w(w+1)\dots(w+n-1)$. To see that F is a solution of the differential equation, set $D = x\frac{d}{dx}$ and notice that if P is a polynomial then

$$P(D)x^n = P(n)x^n.$$

For details, see the notes at the end of this lecture.

The differential equation (3.1) can be written in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

where $p(x)$ and $q(x)$ are rational functions of x having poles only at $x = 0, 1, \infty$. These are therefore the singular points of the differential equation. They are regular singular points because $(x-\xi)p(x)$ and $(x-\xi)^2 q(x)$ are both holomorphic at $x = \xi$ when $\xi = 0$ or 1, and $p(\frac{1}{t})\frac{1}{t}$ and $q(\frac{1}{t^2})\frac{1}{t^2}$ are holomorphic at $t = 0$ (corresponding to $\xi = \infty$).

Part 2: Now suppose that $\Re(a), \Re(c-a) > 0$. Then, the above series has an integral representation

$$F = F(a,b,c;x) = \frac{1}{B(a,c-a)} \int_1^\infty u^{b-c}(u-1)^{c-a-1}(u-x)^{-b}du$$
$$= \frac{1}{B(a,c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}du$$

where for $\Re(\alpha), \Re(\beta) > 0$

$$B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{3.3}$$

To check this integral representation, use the formula

$$(1-ux)^{-b} = \sum_{n=0}^{\infty} \frac{(b,n)}{(1,n)} u^n x^n, \qquad |x| < 1.$$

The hypergeometric series has an analytic continuation outside its circle of convergence $|x| = 1$ to the complex plane minus the segment $[1, \infty)$. This extended function is also called the Gauss hypergeometric function.

Let $\mu_0 = c - b$, $\mu_1 = 1 + a - c$, $\mu_2 = b$, $\mu_3 = 2 - (\mu_0 + \mu_1 + \mu_2)$. Then the conditions $\Re(a), \Re(c-a) > 0$ become $\Re(\mu_1), \Re(\mu_3) < 1$. Imposing the stronger conditions, $0 < \mu_i < 1$ for $i = 0, \ldots, 3$, the μ_i satisfy

$$\sum_{i=0}^{3} \mu_i = 2, \quad 0 < \mu_i < 1, \quad i = 0, \ldots, 3. \tag{3.4}$$

Deligne and Mostow [10] call such a quadruple a ball quadruple. For such a quadruple with $\mu_i \in \mathbb{Q}$, $i = 0, \ldots, 3$, and for $x \in \mathbb{P}_1 \setminus \{0, 1, \infty\}$ the differential form

$$u^{-\mu_0}(u-1)^{-\mu_1}(u-x)^{-\mu_2}du \tag{3.5}$$

is a differential form of the first kind on a smooth projective curve $X(N,x)$ with affine model

$$w^N = u^{N\mu_0}(u-1)^{N\mu_1}(u-x)^{N\mu_2}, \tag{3.6}$$

where N is the least common multiple of the denominators of the μ_i. If $v = \frac{1}{u}$, then

$$u^{-\mu_0}(u-1)^{-\mu_1}(u-x)^{-\mu_2}du = -v^{-\mu_3}(1-v)^{-\mu_1}(1-vx)^{-\mu_2}dv,$$

so μ_3 is the exponent of the integrand at $u = \infty$. Consider the six integrals

$$\int_g^h u^{-\mu_0}(u-1)^{-\mu_1}(u-x)^{-\mu_2}du, \tag{3.7}$$

with $g, h \in \{0, 1, \infty, x\}$. These are the integral representations of six series solutions comprising two independent series solutions at each of the singular points $x = 0, 1, \infty$.

Part 3: If the $(1-\mu_i-\mu_j)^{-1}$ are integers for $i \neq j$ (denoted condition INT), then we can relate the above hypergeometric functions to triangle groups acting discontinuously on one of the simply-connected Riemann surfaces. Namely, let

$$p = |1-\mu_0-\mu_2|^{-1}, \quad q = |1-\mu_1-\mu_2|^{-1}, \quad r = |1-\mu_3-\mu_2|^{-1}. \tag{3.8}$$

By a triangle we mean a region bounded, in spherical geometry by 3 great circles on the Riemann sphere, in Euclidean geometry by 3 straight lines, and in hyperbolic geometry (in the unit disk) by 3 circles orthogonal to the boundary of the unit disk. Consider the triangle with vertex angles π/p, π/q, π/r. Then the relevant geometry depends on the angle sum as follows: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$ (spherical); $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$ (euclidean); $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$ (hyperbolic). Let T denote the interior of a triangle with angles π/p, π/q, π/r and let $\overline{T}$ denote its closure. Then, by Riemann's mapping theorem with boundary, there is a bijective and conformal map $u = u(z)$ of $\mathcal{H}$ onto T which extends continuously to $\mathbb{R}$. Here we view T as a simply-connected subset of $\mathbb{P}_1$ in the spherical case, of $\mathbb{C}$ in the euclidean case and of $\mathcal{H}$ or the unit disk in the hyperbolic case. We denote these simply-connected domains by $\mathcal{D}$. We may assume that u maps $z = 0, 1, \infty$ to the three vertices of $\overline{T}$, with respective angles π/p, π/q, π/r.

This "triangle map" extends across one of the intervals $(\infty, 0), (0,1)$ or $(1,\infty)$ to a biholomorphic map of the lower half plane $\mathcal{H}^-$ onto the image T^- of T by reflection through the corresponding side of $\overline{T}$. Changing the branch of the triangle map by going around any of the points $z = 0, 1, \infty$ changes its value and, by the Schwarz reflection principle, the corresponding images of $\mathcal{H} \cup \mathcal{H}^-$ are non-intersecting copies of $T \cup T^-$.

Now, consider the triangle group $\Delta = \Delta(p,q,r)$ of Möbius transformations, determined up to conjugation by the presentation

$$\langle M_1, M_2, M_3; M_1^p = M_2^q = M_3^r = M_1 M_2 M_3 = \mathrm{Id} \rangle. \tag{3.9}$$

The closure of a fundamental domain for the triangle group is given by $\overline{T} \cup \overline{T^-}$. If a transformation $M_i, i = 1,2,3$ has finite order m, it is conjugate in Δ to a rotation (in the geometry determined by the signature (p,q,r)) through $2\pi/m$ about the vertex of angle π/m of $\overline{T}$. If the order is infinite, then the transformation is conjugate to a translation. The vertices of $\overline{T}$ are, in any case, fixed points of Δ.

The list of spherical and euclidean signatures can be computed directly. The possibilities in the spherical case are: $(2,2,\nu)$, with $2 \leq \nu < \infty$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, and in the euclidean case are: $(2,2,\infty)$, $(2,3,6)$, $(2,4,4)$, $(3,3,3)$.

The successive images of the fundamental domain of $\Delta = \Delta(p,q,r)$ give a tesselation of $\mathcal{D}$ by triangles. In the spherical case, this tesselation consists of twice as many triangles as the order of the corresponding triangle group. In the euclidean and hyperbolic cases this tesselation consists of an infinite number of triangles, and the corresponding triangle groups are also infinite.

Part 4: Consider the differential equation (3.1) for the Gauss hypergeometric function. Let $x_0 \in \mathbb{P}_1 \setminus \{0,1,\infty\}$ and let y_1, y_2 be two linearly independent solutions to (3.1) around x_0. If we analytically continue y_1 and y_2 around a closed curve C in $\mathbb{P}_1 \setminus \{0,1,\infty\}$ which starts and ends at x_0, the functions remain linearly independent solutions. Since y_1 and y_2 span the solution space there is a non-singular matrix

$$M(C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in $\mathbb{C}$ such that y_1 becomes $ay_1 + by_2$ and y_2 becomes $cy_1 + dy_2$ upon analytic continuation around C. A different choice of base point x_0 would yield a matrix in the same $\mathrm{GL}_2(\mathbb{C})$ conjugacy class as $M(C)$. Denoting by $C_1 \circ C_2$ the composition of two closed curves with endpoints at x_0 we have

$$M(C_1 \circ C_2) = M(C_1)M(C_2),$$

with matrix multiplication on the right hand side of this equation. If C_1 can be continuously deformed in $\mathbb{P}_1 \setminus \{0,1,\infty\}$ (with x_0 fixed) into C_2 then $M(C_1) = M(C_2)$. Let $\pi_1(\mathbb{P}_1 \setminus \{0,1,\infty\}, x_0)$ denote the group of homotopy equivalence classes of curves starting and ending at the base point x_0. This is the fundamental group. From the above remarks, we see that we have a homomorphism

$$M : \pi_1(\mathbb{P}_1 \setminus \{0,1,\infty\}, x_0) \to \mathrm{GL}_2(\mathbb{C})$$

called the *monodromy representation* associated to the differential equation. The monodromy group of (3.1) is defined as the image of the monodromy representation. The projective monodromy group of (3.1) is defined as the image of the monodromy group under the natural map $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. If we change the base point x_0 or the choice of basis of the solution space of (3.1), then we conjugate M by an element of $\mathrm{GL}_2(\mathbb{C})$. The conjugacy class of the monodromy group and the projective monodromy group are uniquely determined by (3.1).

The signature (p,q,r) of a triangle group determines its representation, up to conjugacy, in the Möbius transformations. When

$$p = |1-\mu_0-\mu_2|^{-1}, q = |1-\mu_1-\mu_2|^{-1}, r = |1-\mu_3-\mu_2|^{-1},$$

the image of the monodromy representation is $\Delta(p,q,r)$, as given in (3.9).

The case $(p,q,r) = (2,3,\infty)$ corresponds to the elliptic modular group, that is, a representative of the conjugacy class of $\Delta(2,3,\infty)$ is given by $\mathrm{SL}(2,\mathbb{Z})$. It is generated by the two Möbius transformations on $\mathcal{H}$ given by $S : z \mapsto -1/z$ and $T : z \mapsto z+1$. A function j invariant by this group is

$$j(z) = e^{-2\pi i z} + 744 + \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

with the a_n positive integers determined by the first two terms in the above series for j. This is the classical modular function.

Part 5: An arithmetic group is a group commensurable to the integer points $G(\mathcal{O}_F)$ of a linear algebraic group G over a number field F, where $\mathcal{O}_F$ is the ring of integers of F. A linear algebraic group is a group of matrices defined by an algebraic condition. An example is $\mathrm{Sp}(2g,\mathbb{Z})$, an arithmetic subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. As we remarked already, the finite $\Delta(p,q,r)$ were determined by Schwarz. The arithmetic $\Delta(p,q,r)$ were determined by Takeuchi [28]. Takeuchi used the definition of arithmetic Fuchsian group from Lecture I, Part 5. It is not clear *a prioiri* that these two definitions are equivalent, for details see [22]. Certainly the arithmetic groups in the sense of Lecture I are arithmetic in the sense just described. Takeuchi found the following criterion for arithmeticity. Recall that a Fuchsian group (of the first kind) is a group of fractional linear transformations acting properly discontinuously on $\mathcal{H}$ with finite covolume.

Takeuchi's criterion: *Let Γ be a Fuchsian group of the first kind, and let*

$$\mathrm{T} = \{\mathit{trace}(\gamma) : \gamma \in \Gamma\}.$$

Then Γ is arithmetic if and only if
(i)$F = \mathbb{Q}(t)_{(t\in\mathrm{T})}$ is a number field,
(ii)$\mathrm{T} \subseteq \mathcal{O}_F$,
(iii) whenever there is a Galois embedding $\sigma : \mathrm{T} \hookrightarrow \mathbb{R}$ with $\sigma(t^2) \neq t^2$, for some $t \in \mathrm{T}$, then $\sigma(\mathrm{T})$ is a bounded subset of $\mathbb{R}$.

Let Δ be a Fuchsian triangle group of signature (p,q,r). Takeuchi showed that the field F occurring in (i), and generated over $\mathbb{Q}$ by the elements of T, is given by $F = \mathbb{Q}(\cos(\frac{\pi}{p}),\cos(\frac{\pi}{q}),\cos(\frac{\pi}{r}))$. Takeuchi showed that there are, up to permutation, 85 signatures (p,q,r) such that the triangle groups of signature (p,q,r) are arithmetic. He also sorted these into commensurability classes. Therefore, there are infinitely many signatures giving rise to non-arithmetic triangle groups.

Examples: (i) Consider $\Delta(2,5,\infty)$. It is generated by $\begin{pmatrix}1 & \frac{1}{2}(1+\sqrt{5})\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, where $\begin{pmatrix}1 & \frac{1}{2}(1+\sqrt{5})\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(1+\sqrt{5}) & -1\\ 1 & 0\end{pmatrix}$. This last matrix has trace $t = \frac{1}{2}(1+\sqrt{5})$. As $t^2 = \frac{1}{2}(3+\sqrt{5})$, the Galois automorphism of F sending $\sqrt{5}$ to $-\sqrt{5}$ does not leave t^2 fixed. Nonetheless, $\sigma(\mathrm{T})$ is unbounded. Therefore, this group is non-arithmetic.
(ii) We return to the example of Lecture I, Part 4 (see [7]). The parameters $\mu = \{\frac{2}{5},\frac{3}{5},\frac{2}{5},\frac{3}{5}\}$ give $p=5$, $q=\infty$, $r=\infty$, and the triangle group of signature $(5,\infty,\infty)$ is non-arithmetic [28]. It is generated by the two parabolic matrices $\gamma_1 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\gamma_2 = \begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix}$, where $\alpha = (-3+\sqrt{5})/2$, and is contained in the Hilbert modular group of $K = \mathbb{Q}(\sqrt{5})$. This is the group $\Gamma = \mathrm{SL}(2,\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. An element $\gamma \in \Gamma$ acts on $\mathcal{H}^2$ by (γ,γ^σ), and $\Gamma\backslash\mathcal{H}^2$ is the moduli space of a family of abelian varieties with real multiplication.

Notes on Part 1:
To see that the series in (3.2) converges for $|x| < 1$, let $A_n = \frac{(a,n)(b,n)}{(c,n)n!}$. We see that

$$\frac{A_{n+1}}{A_n} = \frac{(a+n)(b+n)}{(c+n)(1+n)}.$$

If a or b is zero or a negative integer then $F(a,b,c;x)$ is a polynomial in x, otherwise

$$\frac{A_{n+1}}{A_n} \mapsto 1$$

as $n \mapsto \infty$ so the series converges for $|x| < 1$ and has $|x| = 1$ as its circle of convergence.
To see that F is a solution of the differential equation (3.1), set $D = x\frac{d}{dx}$ and notice that if P is a polynomial then

$$P(D)x^n = P(n)x^n.$$

Using this and the formula for $\frac{A_{n+1}}{A_n}$, we see that

$$\begin{aligned}
\{(a+D)(b+D) - (c+D)(1+D)x^{-1}\}(\sum_{n=0}^{\infty} A_n x^n) &= \\
\sum_{n=0}^{\infty}\{(a+n)(b+n)A_n x^n - (c+n-1)nA_n x^{n-1}\} &= \\
\sum_{n=0}^{\infty}(a+n)(b+n)A_n x^n - \sum_{n=0}^{\infty}(c+n)(1+n)A_{n+1}x^n &= 0,
\end{aligned}$$

which proves that $F(a,b,c;x)$ satisfies (3.1).

Notes on Part 2: If $c, c-a-b, a-b \notin \mathbb{Z}$, a solution of (3.1) which is linearly independent of $F(a,b,c;x)$, is given in $|x| < 1$ by the series

$$G(x) = x^{1-c}F(a+1-c, b+1-c, 2-c; x).$$

When $\mathrm{Re(c-b)} < 1$ and $\mathrm{Re(b)} < 1$, the series $G(x)$ in the region $|x| < 1$ is equal, up to a constant, to

$$\int_0^x u^{b-c}(u-1)^{c-a-1}(u-x)^{-b}du,$$

as can be seen by a simple computation using the change of variables $u \mapsto \frac{x}{v}$. When one of $c, c-a-b, a-b$ lies in $\mathbb{Z}$, one of the two independent solutions around a regular singular point has a logarithmic singularity at that point. For example, when $c = 1$ the two series solutions F and G coincide. For $c < 1$, the function

$$\frac{1}{1-c}\{x^{1-c}F(a+1-c, b+1-c, 2-c; x) - F(a,b,c;x)\}$$

is a solution of the hypergeometric differential equation. The function given by

$$\lim_{c\to 1}\frac{1}{1-c}\{x^{1-c}F(a+1-c,b+1-c,2-c;x)-F(a,b,c;x)\}$$

is a solution in the case $c=1$ (see [5], pp136-138), and has the form

$$F(a,b,1;x)\log x+F^*(a,b,1;x),$$

where F^* is defined by the equation

$$F^*(a,b,1;x)=\frac{\partial F}{\partial a}+\frac{\partial F}{\partial b}+2\frac{\partial F}{\partial c}.$$

The series expansion of this function in $|x|<1$ is computed in [5], p137.

<u>Notes on Part 3</u>: Schwarz showed that the function $u=u(z)$ of Part 3, that maps $\mathcal{H}$ onto T, satisfies the differential equation

$$\{u,z\}=:\frac{2u'u'''-3u''^2}{2u'^2}=f(z). \tag{3.10}$$

The symbol $\{u,z\}$ is known as the Schwarzian derivative of u with respect to z and is independent of the change of variables

$$u\mapsto\frac{au+b}{cu+d},\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{PSL}(2,\mathbb{C}).$$

Schwarz also calculated the function $f(z)$ explicitly. The result is,

$$f(z)=\frac{1-(\frac{1}{p})^2}{2z^2}+\frac{1-(\frac{1}{q})^2}{2(1-z)^2}+\frac{1-(\frac{1}{p})^2-(\frac{1}{q})^2+(\frac{1}{r})^2}{2z(1-z)}. \tag{3.11}$$

By direct calculation, we find two independent solutions of the classical hypergeometric differential equation (3.1) with $a=\frac{1}{2}(1-\frac{1}{p}-\frac{1}{q}+\frac{1}{r})$, $b=\frac{1}{2}(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r})$, $c=(1-\frac{1}{p})$. The ratio of these solutions satisfies

$$\{u,z\}=\frac{1-(\frac{1}{p})^2}{2z^2}+\frac{1-(\frac{1}{q})^2}{2(1-z)^2}+\frac{1-(\frac{1}{p})^2-(\frac{1}{q})^2+(\frac{1}{r})^2}{2z(1-z)}.$$

Schwarz showed that every regular non-constant solution of this differential equation maps the upper half plane onto a triangle with angles π/p, π/q, π/r.

The Schwarz triangle map is invertible and, by the Schwarz reflection principle, maps $T\cup T^-$ to $\mathcal{H}\cup\mathcal{H}^-$. The function can be analytically continued onto all of $\mathcal{D}$ to yield a mapping

$$j:\mathcal{D}\to\mathbb{P}_1\setminus\{\text{infinite ramification points}\}$$

automorphic with respect to Δ and ramified over 0 with order p (if $p<\infty$) over 1 with order q (if $q<\infty$) and over ∞ with order r (if $r<\infty$).

The case $(p,q,r)=(2,3,\infty)$ corresponds to the elliptic modular group, that is, a representative of the conjugacy class of $\Delta(2,3,\infty)$ is given by $\mathrm{SL}(2,\mathbb{Z})$. It is generated by the two transformations on $\mathcal{H}$ given by $S:z\mapsto$

$-1/z$ and $T : z \mapsto z+1$. The function j has a Fourier expansion which may be normalized as

$$j(z) = e^{-2\pi iz} + 744 + \sum_{n=1}^{\infty} a_n e^{2\pi inz}$$

with the a_n positive integers determined by the first two terms in the above series for j. We recover in this way the classical modular function.

For proofs of most of the results of this section, see [5]. The discussion of the Schwarz triangle maps does not in fact require the INT assumption. For the more general result due to Schwarz see [5], pp.134–135. However, as Schwarz also showed, the INT assumption does ensure that the action of the corresponding triangle group is discontinuous. Knapp [20] lists those hyperbolic triangles whose angles are not of the form $\pi/p, \pi/q, \pi/r$, for $2 \leq p, q, r \leq \infty$ positive integers, but whose triangle groups are nonetheless discrete in $\mathrm{PSL}(2, \mathbb{R})$.

4. Lecture IV: Transcendence of Special Values of Hypergeometric Functions

Part 1: A recurrent theme in transcendental numbers theory is the study of the set of algebraic points at which a transcendental function takes algebraic values. We call this the *exceptional set* of the function. The classical work of Hermite and Lindemann shows that the exceptional set of the exponential function $\exp(x)$, $x \in \mathbb{C}$, consists only of $x = 0$. This implies that e and π are transcendental numbers. C.L. Siegel (1929) suggested studying the exceptional set of the classical hypergeometric function $F = F(a, b, c; x)$, when a, b, c are rational, and asked whether this set is finite. Wolfart [30] showed that this exceptional set corresponds to a subset of the CM (complex multiplication) points on a moduli space of abelian varieties associated to a, b, c. He gave sufficient conditions on a, b, c for the exceptional set to be infinite. Next, P.B. Cohen (me!) and Wüstholz [8] gave sufficient conditions on a, b, c for the exceptional set to be finite. Our proof, however, assumed the validity of a particular case of the André–Oort conjecture, which was subsequently established in a deep paper by Edixhoven and Yafaev in [18].

The exceptional set of F is $\mathcal{E} = \{x \in \overline{\mathbb{Q}} : F(a, b, c; x) \in \overline{\mathbb{Q}}^*\}$. We can assume the monodromy group of F is infinite, otherwise F is algebraic and $\mathcal{E} = \overline{\mathbb{Q}}$. As we have seen, the function F has an integral representation,

$$F(a, b, c; x) = C \int_\gamma u^{b-c}(u-1)^{c-a-1}(u-x)^{-b} du \Big/ \int_\gamma u^{-c}(u-1)^{c-a-1} du$$

where γ is a Pochhammer cycle around 1, ∞ and $C \in \overline{\mathbb{Q}}^*$. The numerator is a period on the curve $X_\mu(x)$ of Lecture I with affine model

$$w^N = u^{N\mu_0}(u-1)^{N\mu_1}(u-x)^{N\mu_2},$$

where N is the least common denominator of the exponents $\mu_0 = c - b$, $\mu_1 = a + 1 - c$, $\mu_2 = b$. The denominator is a period on $X_\mu(0)$. Assume, again for simplicity, that $N = p$, prime, and that the differential form $du/w(x)$ in the integrand of the above formula is holomorphic, even at $x = 0$. This will follow from the conditions $0 < \mu_i < 1$, $i = 0, 1, 2, 3$, $\mu_0 + \mu_2 < 1$. Recall from Lecture I that, for $x \neq 0, 1, \infty$, the genus of $X_\mu(x)$ is $p - 1$, and the genus of $X_\mu(0)$ is $(p-1)/2$. There is a natural action of the p-th roots of unity on these curves, given by $(u, w) \mapsto (u, \zeta_p^{-1} w)$, for $\zeta_p = \exp(2\pi i/p)$. Therefore $K = \mathbb{Q}(\zeta_p) \subseteq \mathrm{End}_0(\mathrm{Jac}(X_\mu(x)))$. In fact, $\mathrm{Jac}(X_\mu(0))$ has CM. When x is in $\mathcal{E}$, the numerator and denominator of the integral expression for F are proportional by a constant in $\overline{\mathbb{Q}}^*$. Applying Theorem 2.4, we have $\mathrm{Jac}(X(x)) \widehat{=} \mathrm{Jac}(X(0)) \times A'$. Now $K \subseteq \mathrm{End}_0(A')$ with $\dim(A') = (p-1)/2$, and A' has CM. Therefore, if $x \in \mathcal{E}$, then $\mathrm{Jac}(X(x))$ has CM and is isogenous to $\mathrm{Jac}(X(0)) \times A'$. The converse is also true. Now $\mathrm{Jac}(X(0))$ and A' are not necessarily simple. They may be powers of an abelian variety with CM by a subfield of K. There are easily applicable criteria for the "simplicity" of a CM type, see for example [21].

Part 2: The map $\varphi : \mathbb{P}_1 \setminus \{0,1,\infty\} \to \mathcal{A}_{p-1}$, mapping x to the point of the Siegel modular variety representing the isomorphism class of the abelian variety $\mathrm{Jac}(X_\mu(x))$, is an example of a *period map*. The analytic space $\mathcal{A}_{p-1}$ can be compactified to a projective variety, and the closure of the image of φ in the strong topology has the structure of an irreducible algebraic curve. Let Z be the intersection of this curve with $\mathcal{A}_{p-1}$. Let Ω be the set of points of $\mathcal{A}_{p-1}$ representing isomorphism classes of abelian varieties isogenous to $\mathrm{Jac}(X(0))\times A'$. Then $\Omega \subset V_{p-1}(\overline{\mathbb{Q}})$, where, as before, V_{p-1} is the underlying quasi-projective variety of $\mathcal{A}_{p-1}$. By the above results, the exceptional set is now $Z \cap \Omega$.

As we mentioned in Lecture I, $\mathcal{A}_{p-1}$ is not the smallest Shimura variety containing Z. We can think of a Shimura variety as being the quotient of a complex symmetric domain by an arithmetic group which is also the moduli space for a family of abelian varieties with some specified structure. The smallest Shimura variety S_Z with $Z \subseteq S_Z$ will correspond to the smallest arithmetic group Γ_0 containing the monodromy group $\Gamma(a,b,c) = \Gamma_\mu$ of F. Let k be the field generated by the traces of the elements of Γ_μ. This is a totally real number field of degree n over $\mathbb{Q}$. The group algebra $k[\Gamma_\mu]$ in $M_2(\mathbb{R})$ is a quaternion algebra A over k and $\mathcal{O}_k[\Gamma_\mu]$ is an order in A. See, Lecture I, Part 5, to recall the definitions used here. The norm unit group $\Gamma = \Gamma(A, \mathcal{O}_k)$ of elements of this order of reduced norm 1 is an arithmetic group containing Γ_μ. This generalizes the arithmetic group of Lecture I, Part 5. Sometimes $\Gamma_0 \subsetneq \Gamma$ with finite index. It is easy to reduce to the case $\Gamma_0 = \Gamma$. There is an $\mathbb{R}$-algebra isomorphism from $A \otimes_{\mathbb{Q}} \mathbb{R}$ to $M_2(\mathbb{R})^t \oplus \mathbb{H}^{n-t}$, where t is the number of unramified infinite places of k for A. In the language of Lecture I, Part 5, this means the number of Galois embeddings σ of k into $\mathbb{R}$ for which $A^\sigma \otimes \mathbb{R}$ is $M_2(\mathbb{R})$. This isomorphism defines an action of Γ on $\mathcal{H}^t$. In Lecture I, Part 5, we had $t = 1$. The quotient $S = \Gamma \backslash \mathcal{H}^t$ is a Shimura variety. It parameterizes the isomorphism classes of abelian varieties of dimension $2n$ with generalized complex multiplication of type Φ by an order in a purely imaginary quadratic extension L of k. The field L is a subfield of $K = \mathbb{Q}(\exp(2\pi i/p))$. The type Φ is determined by the ramification of A at the infinite places of k, see Lecture I.

Example: In Example (ii) of Lecture 3, Part 5, $t = 2$ and $k = \mathbb{Q}(\sqrt{5})$. The group Γ_μ is the triangle group $(5, \infty, \infty)$ and the arithmetic group Γ is the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_k)$. Recall that $\gamma \in \Gamma$ acts on $\mathcal{H}^2$ by the Möbius transformation (γ, γ^σ), where $\sigma(\sqrt{5}) = -\sqrt{5}$. The Hilbert–Shimura variety $\Gamma \backslash \mathcal{H}^2$ is a moduli space for isomorphism classes of (polarized) abelian surfaces B with real multiplication by $\mathcal{O}_k \subseteq \mathrm{End}(B)$. The Jacobian $J_\mu(x)$, $x \neq 0, 1$ of the Example of Lecture I, Part 4, is isogenous to the square of such an abelian surface $B(x)$. The Jacobian $J_\mu(0)$ is a simple abelian surface with complex multiplication by $\mathbb{Q}(\zeta_5)$ of type $\{\sigma_1, \sigma_3\}$. When $x \in \mathcal{E}$, $J_\mu(x)$ is isogenous to $J_\mu(0) \times A'$, where A' is a simple abelian surface with CM by $\mathbb{Q}(\zeta_5)$ and type $\{\sigma_2, \sigma_4\} = \{\sigma_{-1}, \sigma_{-3}\}$. The generalized CM type of $J_\mu(x)$, $x \neq 0, 1$ is $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} = \{\sigma_1, \sigma_3\} \cup \{\sigma_2, \sigma_4\}$.

Part 3: The results on the exceptional set of the hypergeometric function, due to Wolfart, Wüstholz, myself, Edixhoven and Yafaev, can be stated as follows.

Theorem 4.1. *Suppose $a, b, c \in \mathbb{Q}$ and $\Gamma(a, b, c)$ is infinite. The exceptional set*

$$\mathcal{E} = \{z \in \overline{\mathbb{Q}} : F(a, b, c; z) \in \overline{\mathbb{Q}}\}$$

is infinite if and only if $\Gamma(a, b, c)$ is arithmetic. Moreover, every element of the set $\mathcal{E}$ corresponds to a CM point in a certain moduli variety $V(a, b, c)$ for abelian varieties, also known as a Shimura variety.

Recall that $\mathcal{E}$ corresponds to the points of $S = \Gamma\backslash\mathcal{H}^t$ whose corresponding abelian varieties lie in the same isogeny class as a fixed abelian variety with CM. The André–Oort Conjecture predicts the following.

Conjecture 4.2. *Let $Z \subseteq S$ be an irreducible algebraic curve in a Shimura variety S. The intersection of Z with the CM points of S is infinite if and only if Z is the quotient of a complex symmetric domain by an arithmetic group.*

This conjecture follows from GRH (Klingler, Ullmo, Yafaev). A deep unconditional result [18] of Edixhoven–Yafaev is as follows.

Theorem 4.3. *Let $Z \subseteq S$ be an irreducible algebraic curve in a Shimura variety S. The intersection of Z with the CM points of S in a fixed isogeny class is infinite if and only if Z is the quotient of a complex symmetric domain by an arithmetic group.*

Therefore, the set $\mathcal{E}$ will be infinite if and only if $\Gamma_\mu\backslash\mathcal{H}$ is itself a Shimura curve, that is, if and only if the monodromy group is arithmetic.

EXAMPLE: (i) The triangle group $(5, \infty, \infty)$ is not maximal. It is contained in the so-called *Hecke group* of index 10, that is, the triangle group of order $(2, 10, \infty)$. Now $X_\mu(x)$, $x \neq 0, 1$ has affine model $w^{10} = u^7(u-1)^8(u-x)^2$. The least common denominator of the μ's is 10, which is not prime, but the methods are still applicable. It turns out that the triangle group is still contained in $\mathrm{SL}(2, \mathcal{O}_k)$, for $k = \mathbb{Q}(\sqrt{5})$.

(ii) The Hecke triangle group $(2, 5, \infty)$ is also a non-arithmetic of $\mathrm{SL}(2, \mathcal{O}_k)$, $k = \mathbb{Q}(\sqrt{5})$. Now $X_\mu(x)$, $x \neq 0, 1$ has model $w^{20} = u^{13}(u-1)^{17}(u-x)^3$, and the trace field actually leads to the larger field $\mathbb{Q}(\cos(\pi/10))$. As 20 is not prime, we have to consider a factor $T_\mu(x)$ of $J_\mu(x)$ of dimension $8 = |(\mathbb{Z}/20\mathbb{Z})^*|$. It turns out that $T_\mu(x)$ is isogenous to the 4th power of an abelian surface with real multiplication by $\mathbb{Q}(\sqrt{5})$ ([7]). The corresponding hypergeometric function is $F(\frac{13}{20}, \frac{3}{20}, \frac{4}{5}, x)$. By the above results, the set $\mathcal{E}$ for this function is finite, and its elements correspond to certain abelian surfaces with CM by an imaginary quadratic extension of $\mathbb{Q}(\sqrt{5})$.

(iii) $\mathrm{SL}(2, \mathbb{Z})$ is the arithmetic triangle group $(2, 3, \infty)$ and corresponds to $F(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}, x)$, which has infinite $\mathcal{E}$. Beukers–Wolfart give explicit values of $F(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}, x)$ at certain elements of $\mathcal{E}$ [3].

Part 4: Research initiated recently [16], [12], [13], [15] extends some of these results to functions of several complex variables. Here, the Appell-Lauricella hypergeometric functions of several complex variables replace the Gauss hypergeometric function of one variable. Once again, there is a family of abelian varieties associated to an Appell-Lauricella function and a finite to one morphism to its base space (Shimura variety) from the domain of the Appell-Lauricella function. At this point, an important distinction between functions of one and several complex variables manifests itself. Namely, the points in the exceptional set of an Appell-Lauricella function do not necessarily correspond to abelian varieties with CM as they did in the one variable case. Moreover, the property that replaces finiteness in the one variable case is that the exceptional set not be Zariski dense.

Let us now focus, for simplicity, on the two variable case. The two variable Appell hypergeometric series is given by

$$F = F(x,y) = F(a,b,b',c;x,y) = \sum_{m,n} \lambda_{m,n} x^m y^n, \qquad |x|,|y| < 1,$$

where we assume $a,b,b',c \in \mathbb{Q}$. The $\lambda_{m,n}$ are the rational numbers

$$\lambda_{m,n} = \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)(1,m)(1,n)},$$

where $(w,n) = w(w+1)\dots(w+n-1)$.

The Appell function F is the solution of a system of second order p.d.e's with regular singularities along the lines $x,y = 0,1,\infty$ and $x = y$. Its analytic continuation is given by the Euler integral representation

$$F(x,y) = \int_1^\infty \frac{du}{w(x,y)} \Big/ \int_1^\infty \frac{du}{w(0,0)},$$

where $w = w(x,y)$ is the algebraic function of u given by

$$w(x,y)^N = u^{N(c-b)}(u-1)^{N(1+a-c)}(u-x)^{Nb}(u-y)^{Nb'}.$$

Here N is the least common denominator of $c-b$, $1+a-c$, b, b'. The Riemann surface $X(x,y)$ of w is a smooth curve defined over $\overline{\mathbb{Q}}(x,y)$. Assume that a,b,b',c are such that $du/w(x,y)$ is holomorphic, even at $(x,y) = (0,0)$. The exceptional set of F is given by

$$\mathcal{E} = \{(x,y) \in \overline{\mathbb{Q}}^2 : F(x,y) \in \overline{\mathbb{Q}}^*\}$$

Theorem 4.4. *The exceptional set of F is Zariski dense in $\mathbb{P}_1 \times \mathbb{P}_1$ if and "only if" the monodromy group of F is arithmetic in $PU(1,2)$.*

There is a finite list of a,b,b',c giving arithmetic lattices [10]. For the proof, we consider the Jacobian $\mathrm{Jac}(X(x,y))$. To simplify, suppose $N = p$, prime. Then, $g(X(x,y)) = 3(p-1)/2$ and $g(X(0,0)) = (p-1)/2$, where g is the genus. The automorphism $(u,w) \to (u, \zeta^{-1}w)$, $\zeta = \exp(2\pi i/p)$ of $X(x,y)$ induces an embedding of $\mathbb{Q}(\zeta)$ inside $\mathrm{End}_0(\mathrm{Jac}(X(x,y)))$. It turns out that $\mathrm{Jac}(X(0,0))$ is even a CM abelian variety.

Part 5: For (x,y) in the exceptional set, $X(x,y)$ is defined over $\overline{\mathbb{Q}}$ and

$$\int_1^\infty \frac{du}{w(x,y)} = \alpha \int_1^\infty \frac{du}{w(0,0)},$$

for some $\alpha \in \overline{\mathbb{Q}}^*$. We then use transcendence techniques to deduce from this that there is an isogeny (surjection with finite kernel) of the form

$$\mathrm{Jac}(X(x,y)) \widehat{=} \mathrm{Jac}(X(0,0)) \times A(x,y).$$

Recall that $\mathrm{Jac}(X(0,0))$ has CM, but we only know that

$$\dim(A) = \dim(A(x,y)) = p-1,$$

and

$$\mathbb{Q}(\zeta) \subseteq \mathrm{End}_0(A(x,y)).$$

For example, this abelian variety is not necessarily CM. This introduces an additional difficulty compared to the one variable case, where the exceptional set corresponds only to CM abelian varieties.

From the association $(x,y) \to \mathrm{Jac}(X(x,y))$, we get a period map with two dimensional image

$$p : \mathbb{C}^2 \to S$$

where S is the smallest moduli space of abelian varieties containing $p(\mathbb{C}^2)$. From the association $(x,y) \to A(x,y)$, we have a map

$$q : \mathbb{C}^2 \to S' \subseteq S$$

where S' is a smaller moduli space of abelian varieties. The exceptional set of F corresponds to $p(\overline{\mathbb{Q}}^2) \cap q(\overline{\mathbb{Q}}^2)$. This leads to asking,

Let Z be the Zariski closure in S of the image of the period map p. When is the intersection of Z with (the Hecke orbit of) S' Zariski dense in Z?

For arithmetic monodromy, we have $\dim(S) = 2$ and Zariski density. Otherwise $\dim(S) > 2$ and we can show that $Z \cap \mathrm{orbit}(S')$ is not Zariski dense in Z if *the intersection of Z with the union of all Shimura subvarieties of S of dimension strictly less than* $\dim(S) - 2$ *is not Zariski dense in Z.*

Such a condition follows, for example, from more general conjectures of Pink and Zilber on "mixed" Shimura varieties. In the case of the "pure" Shimura variety given by the Siegel modular variety $\mathcal{A}_g$ of genus g, it asks the following. If $Z \subseteq \mathcal{A}_g$ is a subvariety of the Siegel modular variety, we let S_Z be the smallest special subvariety of $\mathcal{A}_g$ containing Z.

Conjecture 4.5. *Let $Z \subseteq \mathcal{A}_g$ be an irreducible algebraic subvariety. Then the intersection of Z with the union of all special subvarieties of dimension strictly less than* $\dim(S_Z) - \dim(Z)$ *is not Zariski dense in Z.*

The Shimura subvarieties of dimension zero are the CM points. Pink's conjecture says that if Z is not a special subvariety, then the set of CM points on Z is not Zariski dense in Z, which is the André–Oort conjecture.

Notes on Part 5: The analogues of Conjecture 4.5 for multiplicative tori $\mathbb{G}_m^r$ and for abelian varieties raise interesting diophantine problems. These questions can be put into a single framework using the language of mixed Shimura varieties. The CM points on Shimura varieties are analogous to torsion points, or points of finite order, on an abelian variety or a torus. We have an understanding of the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on both types of points. Moreover, the torsion points of an abelian variety or a torus form a subgroup, and the set of CM points are invariant under the correspondences defined by $\mathrm{Sp}(2g, \mathbb{Q})$. A semi-abelian variety is an extension of an abelian variety A by a torus $\mathbb{G}_m^r$, and therefore A and $\mathbb{G}_m^r$ are themselves semi-abelian varieties.

For a semi-abelian variety A and an integer $d \geq 0$, let $A^{[d]}$ be the union of all algebraic subgroups of A of codimension strictly greater than d. The conjectures of Zilber and Pink give in this case the following.

Conjecture 4.6. *Let A be a complex semi-abelian variety. Let $X \subseteq A$ be an irreducible closed subvariety of dimension d that is not contained in any proper algebraic subgroup of A. Then $X \cap A^{[d]}$ is not Zariski dense in X.*

When A is an abelian variety, the Manin-Mumford conjecture states that $X \cap A_{\mathrm{tors}}$ is not Zariski dense in X. This was proved by Raynaud in 1983. Other partial results in the abelian case, mainly when $\dim(X) = 1$ are due to Ratazzi, Rémond–Viada. When $\dim(X) = 1$ and $A = \mathbb{G}_m^r$, the conjecture is due to Bombieri–Masser–Zannier [4], who proved it for $r \leq 5$. It was proved for all $r \geq 1$ by Maurin. Therefore, if X is not contained in a proper algebraic subgroup of $\mathbb{G}_m^r$, then X contains only finitely many points in the union of algebraic subgroups of codimension at least 2. Bombieri–Masser–Zannier also showed that, over $\overline{\mathbb{Q}}$, if $X \subset \mathbb{G}_m^r$ is not contained in a translate of a proper algebraic subgroup, then the points of X lying in a proper algebraic subgroup have bounded height. Recall that, roughly speaking, the height of an algebraic number measures the size of its numerator and denominator, or its arithmetic complexity.

Consider, for example, the curve $X \subset \mathbb{G}_m^2$ given by $x + y = 1$. Then, the points of X lying in a proper algebraic subgroup are precisely those x such that x and $1 - x$ are multiplicatively dependent. That is, there are integers m, n, not both zero, with $x^m(1-x)^n = 1$. By completely different methods to those of Bombieri–Masser–Zannier, we obtained with Zannier [9] a much finer height bound in this case. Namely, we showed that if x and $1 - x$ are multiplicatively dependent algebraic numbers, then $M(x) = \max(H(x), H(1-x)) \leq 2$, with the bound being attained for $x = 1/2, 2$. Moreover, the value 2 is an isolated point in the range of M. Habegger [19] generalized this result to curves $x + y = \alpha$ in $\mathbb{G}_m^2$, where $\alpha \in \overline{\mathbb{Q}}^*$, showing for example that $H(x, y) \leq 2H(\alpha)$ if $\alpha \in \mathbb{Z}$, with the upper bound being attained and isolated if and only if α is a power of 2.

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