TRANSCENDENCE IN POSITIVE CHARACTERISTIC

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Contents

1. Table of symbols

p	$:=$	a fixed prime
$\mathbb{F}_q$	$:=$	finite field of $q = p^m$ elements
A	$:=$	$\mathbb{F}_q[\theta]$ = polynomials in the variable θ
k	$:=$	$\mathbb{F}_q(\theta)$ = rational functions in θ
k_∞	$:=$	$\mathbb{F}_q(\!(1/\theta)\!)$ = ∞-adic completion of k
$\lvert\cdot\rvert_\infty$	$:=$	absolute value on $\tilde{k}$ such that $\lvert\theta\rvert_\infty = q$
$\overline{k_\infty}$	$:=$	algebraic closure of k_∞
ζ_θ	$:=$	a fixed $(q-1)$-st root of $-\theta$ in $\overline{k_\infty}$
$\mathbb{C}_\infty$	$:=$	completion of $\overline{k_\infty}$ w.r.t. $\lvert\cdot\rvert_\infty$
$\overline{k}$	$:=$	algebraic closure of k in $\mathbb{C}_\infty$
A_+	$:=$	the set of elements of A monic in θ
deg	$:=$	the function associating to each element of A its degree in θ
F	$:=$	the q-power Frobenius map sending $x \mapsto x^q$
$c^{(i)}$	$:=$	c^{q^i} = the ith iterate of F applied to an element c of $\mathbb{C}_\infty$
$\mathbb{C}_\infty\{F\}$	$:=$	the ring of *twisted polynomials* $\sum_i a_i F^i$, where multiplication is given by $a_i F^i(a_j F^j) = a_i a_j^{q^i} F^{i+j}$

2. Transcendence for Drinfeld modules

2.1. Wade's results.

2.1.1. *Wade's results.* The story of transcendence over function fields begins with Wade's proof of the non-algebraicity (over k) of the power series

$$\pi_q = \theta\zeta_\theta \prod_{i=1}^{\infty}\left(1 - \theta^{1-q^i}\right)^{-1}.$$

The general mathematical community had little idea why Carlitz [Carlitz 1935] considered this to be the function field analogue of the classical number π.

The situation became clearer through the work of V. G. Drinfeld [Drinfeld 1974] and D. Hayes [Hayes]. Drinfeld introduced what he called *elliptic modules*, now commonly called *Drinfeld modules*, because they have strikingly similar similarities with classical elliptic curves. To minimize notation I will not speak of the most general type of Drinfeld modules. For that and details omitted here, see [Goss], [Rosen], or [Thakur]. (Added by MP: A great source for information in these notes is [Brownawell 1998].)

2.2. **Drinfeld modules.** What Drinfeld discovered is that, just as two-dimensional lattices in the complex numbers correspond to elliptic curves, (arbitrary rank) lattices in $\mathbb{C}_\infty$ over A correspond to representations

$$\phi\colon A \to \mathbb{C}_\infty\{F\}$$

for which $\phi(\theta) = \theta F^0 +$higher terms, and the analogy is amazingly tight, including analogues of the Weierstrass elliptic functions

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda}{}' \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

2.3. **The Weierstraß-Drinfeld correspondence.** Here is the correspondence, where by *lattice* we mean a discrete free sub-A-module of $\mathbb{C}_\infty$ of finite rank.

Weierstraß	Drinfeld
$\mathbb{Z}$	A
2-dim'l lattice Λ	r-dim'l A-lattice Λ
$\exp_\Lambda(z) = (\wp(z), \wp'(z))$	$\exp_\Lambda(z) := z \prod_{\lambda\in\Lambda}{}'(1 - \frac{z}{\lambda})$ analytic!
elliptic curve $\mathcal{E}\colon y^2 = 4x^3 - g_2x - g_3$	$\mathbb{G}_a$
$0 \to \Lambda \to \mathbb{C} \to \mathcal{E}_\lambda(\mathbb{C}) \to 0$	$0 \to \Lambda \to \mathbb{C}_\infty \to \mathbb{G}_a(\mathbb{C}_\infty) \to 0$
Isogenies from c s.t. $c\Lambda_1 \subset \Lambda_2$	Isogenies from c s.t. $c\Lambda_1 \subset \Lambda_2$ when $\operatorname{rank}_A \lambda_1 = \operatorname{rank}_A \Lambda_2$
$\mathbb{Z} \subseteq \operatorname{End}(\mathcal{E})$	$\phi_\Lambda\colon A \to \operatorname{End}\mathbb{G}_a$ via $\phi_\Lambda(\theta) = \theta F^0 +$ higher terms $\in \mathbb{C}_\infty\{F\}$

2.3.1. *Drinfeld exponential function.* Convergence of the infinite product $\exp_\lambda(z)$ for all values of z follows immediately from the discreteness of Λ: Only finitely many $\lambda \in \Lambda$ lie within any given distance to the origin.

2.3.2. *Exercise.* For each N, consider the truncated product

$$e_d(z) := z \prod{}'' \left(1 - \frac{z}{\lambda}\right),$$

where the $''$ indicates that, for a fixed basis $\lambda_1, \ldots, \lambda_r$ for Λ, the product is taken over all non-zero λ involving coefficients from A with degrees at most d. Show that

$$e_d(z) = z + a_1 z^q + a_2 z^{q^2} + \cdots + a_n z^{q^n},$$

with $a_1, \ldots, a_n \in k(\lambda_1, \ldots, \lambda_r)$, where $n = q^{r(d+1)}$. Hint: Show that the coefficients of e_d must be invariant under multiplication by elements from $\mathbb{F}_q$.

2.3.3. $\mathbb{F}_q$*-linearity of* $\exp_\Lambda(z)$. As a consequence of the preceding exercise, it follows that the Drinfeld exponential function has an expansion of the form

$$\exp_\lambda(z) = \sum_{n=0}^{\infty} a_n z^{q^n},$$

with $a_0 = 1$, i.e. it is an $\mathbb{F}_q$-*linear* power series. Thus, for $c \in \mathbb{F}_q$,

$$\exp_\lambda(z_1 + cz_2) = \exp(z_1) + c\exp_\Lambda(z_2).$$

Moreover $\exp_\lambda(z)$ visibly parametrizes $\mathbb{G}_a(\mathbb{C}_\infty)$ and has kernel Λ. So the sequence

$$0 \to \Lambda \to \mathbb{C}_\infty \xrightarrow{\exp_\Lambda} \mathbb{C}_\infty \to 0$$

is exact.

2.3.4. *Lattice containment.* First consider the case that Λ_1, Λ_2 are A-lattices of the same A-rank, but $\Lambda_2 \subseteq \Lambda_1$. Then Λ_1/Λ_2 is a finite dimensional $\mathbb{F}_q$-vector space, say with coset representatives $\lambda_1 (= 0), \dots, \lambda_d$. Then

$$P_{[\Lambda_2\colon \Lambda_1]}(X) := X \prod{}' \left(1 - \frac{X}{\exp_{\Lambda_2}(\lambda_i)}\right)$$

is an $\mathbb{F}_q$-linear polynomial with X as lowest term, and it provides the crucial functional relation

$$\exp_{\Lambda_1}(z) = P_{[\Lambda_2\colon \Lambda_1]}(\exp_{\Lambda_2}(z)),$$

since both sides have the same zeros and the same leading terms.

2.3.5. *Lattice morphism.* Let $c \in \mathbb{C}_\infty$ be non-zero. Now the product expansion for $\exp_\Lambda(z)$ makes obvious that

$$\exp_{c\Lambda}(cz) = c\exp_\Lambda(z).$$

When in additiona Λ_1, Λ_2 have the same rank and $c\Lambda_2 \subseteq \Lambda_1$, then $[c\Lambda_2\colon \Lambda_1]$ is finite. Then

$$\exp_{\Lambda_1}(cz) = P_{[c\Lambda_2\colon \Lambda_1]}(\exp_{c\Lambda_2}(cz)) = P_{[c\Lambda_2\colon \Lambda_1]}(c\exp_{\Lambda_2}(z))$$

as both sides have the same zeros and the same leading terms. Writing $P_{[c\Lambda_2\colon \Lambda_1]}(cz) = \psi z$ for $\psi = cF^0 +$ higher terms in $\mathbb{C}_\infty\{F\}$ and replacing z in this relation by az, $a \in A$ gives

$$\phi_1(a)\psi\exp_{\Lambda_2}(z) = \phi_1(a)\exp_{\Lambda_1}(cz) = \exp_{\Lambda_1}(acz) = \psi\exp_{\Lambda_2}(az) = \psi\phi_2(a)\exp_{\Lambda_2}(z)$$

We say that $\psi \in \mathrm{Hom}_A(\phi_1, \phi_2)$.

2.3.6. *Isogenies.* Since $\exp_{\Lambda_2}(z)$ is a transcendental function (it has infinitely many zeros), we conclude that

$$\phi_{\Lambda_1}(a)\psi = \psi\phi_{\Lambda_2}(a).$$

Any non-zero element of $\mathbb{C}_\infty\{F\}$, the *twisted polynomials over* $\mathbb{C}_\infty$, satisfying this property will be called an *isogeny* from ϕ_{Λ_2} to ϕ_{Λ_1}, and we write

$$\psi\colon \phi_{\Lambda_2} \to \phi_{\Lambda_1}.$$

Note that, as $c \neq 0$, $\psi = cF^0 +$ higher terms $\neq 0$ in the previous displayed line.

2.3.7. *Exercise.* Show that if $\psi\colon \phi_{\Lambda_1} \to \phi_{\Lambda_2}$ is an isogeny, then there is another isogeny $\widehat{\psi}\colon \phi_{\Lambda_2} \to \phi_{\Lambda_1}$. Hint: Use the fact that ψ comes from a non-zero c with $c\Lambda_1 \subset \Lambda_2$.

2.3.8. *Analytic uniformization.* Drinfeld noted that these considerations are reversible!

Theorem 2.3.9 (Drinfeld's Uniformization Theorem)**.** *Given a homomorphism $\phi : A \to \mathbb{C}_\infty\{F\}$ such that $\phi(\theta) = \theta F^0 + \cdots + a_m F^m$, $a_m \neq 0$, $m > 0$, there is a unique A-lattice Λ such that $\phi = \phi_\Lambda$. Moreover* $\operatorname{rank}_A \Lambda = m$.

The way to see this is to use the condition $e(\theta z) = \phi(\theta)e(z)$ to define a unique $\mathbb{F}_q$-linear power series $e(z)$ with leading term z. Then one shows that $e(z)$ is $\mathbb{F}_q$-linear and entire. Finally, from the functional equation, one sees that the zeros of $e(z)$ form a discrete A-module, i.e. a lattice Λ. Then show that $e(z) = \exp_\Lambda(z)$. For a complete proof, see [Goss], [Rosen], [Thakur].

2.3.10. *Exercise.* Show that if $\psi\colon \phi_{\Lambda_2} \to \phi_{\Lambda_1}$ and $\psi = cF^0+$ higher terms, then $c \neq 0$ and $c\Lambda_2 \subseteq \Lambda_1$. Hint: Show that $\psi \exp_{\Lambda_2}(z/c)$ satisfies the functional equation for $\exp_{\Lambda_1}(z)$ and has lowest term z.

Nothing surprising occurs here, but there are some details to verify. This is the function field analogue of one of the major settings of modern transcendence theory. The first major steps in its development were taken almost single-handedly by one of our speakers, M. Waldschmidt.

Here we have the analogue of a commutative agebraic group $\mathbb{G}_a$ with exponential function:

$$0 \to \Lambda \to \mathrm{Lie}_G \to G(\mathbb{C}_\infty) \to 0$$

2.3.11. *Drinfeld module $\mathbb{G}_a$.* If $\phi(\theta) = \theta F^0$, then we can consider $\exp(z) = z$ with period $\Lambda = \{0\}$.

2.3.12. *Field of definition.* If K is a field, $k \subseteq K \subseteq \mathbb{C}_\infty$ and ϕ_Λ is a Drinfeld module, we will say that ϕ is *defined over* K whenever either (and therefore both) of the two equivalent conditions hold:

- $\phi(\theta) \in K\{F\}$
- $\exp_\Lambda(z) \in K[\![z]\!]$.

The conditions are equivalent because of the functional equation.

2.4. **Carlitz.**

2.4.1. *Carlitz exponential function.* What does this have to do with Carlitz? Carlitz defined an explicit analytic exponential function, which we will denote by $e_C(z)$:

$$e_C(z) := \sum_{h \geq 0} \frac{z^{q^h}}{D_h}$$

where the D_h are defined recursively by $D_0 := 1, \quad D_h = (\theta^{(i)} - \theta)D_{i-1}^{(1)}$. Moreover

$$\exp_C(\theta z) = \theta \exp_C(z) + (\exp_C(z))^q = (\theta F^0 + F)\exp_C(z) \qquad (2.4.1.1)$$

for all $z \in \mathbb{C}_\infty$. That is $\phi_C(\theta) = \theta F^0 + F$.

2.4.2. *Easy exercise.* Verify this functional equation.

2.4.3. *Carlitz period.* Drinfeld tells us (and Carlitz showed) that there is an underlying lattice Λ_C for which $\exp_C = \exp_\Lambda$, and that it has rank one. In fact, Carlitz found a basis for his Λ, namely $\lambda_C = A\ \sqrt[q-1]{\theta - \theta^{(1)}}\pi_q$. That is, if $\exp_C(z)$ is the correct analogue of the ordinary exponential function, then since π_q lies in the completion of k with respect to the absolute value $|\cdot|_\infty$ and $\sqrt[q-1]{\theta - \theta^{(1)}} \in \overline{k}$, these seem to be the correct analogues of π and $2i$.

2.4.4. *Analogue of exponential function?* But why should $\exp_C$, rather than some $\exp_\lambda$ with $\Lambda \neq \Lambda_C$, be the correct analogue of the ordinary exponential function? After all, the rank one Drinfeld modules are all isogenous. Hayes provides the first major answer (that Carlitz knew), namely the function field analogue of the classical Kronecker-Weber theorem.

Theorem 2.4.5 ([Hayes])**.** *The Hilbert class fields over k are generated (in an explicit manner) by the Carlitz torsion values $\exp_C(\frac{\lambda}{a})$, $a \in A$, $\lambda \in \Lambda_C$.*

2.4.6. *Gelfond-Schneider-Wade.* So Carlitz was right, and now we see why Wade's result above is considered the analogue of Hermite's theorem on the transcendence of π.

Maybe we shouldn't feel too bad. It wasn't until Wade's third paper that he adapted a classical proof of transcendence to the function field setting. There he established the following:

Theorem 2.4.7 (Wade's Analogue of Gelfond-Schneider [Wade 1946])**.** *Let $\alpha \in \mathbb{C}_\infty$ be non-zero with $\exp_C(\alpha) \in \overline{k}$ and let $\beta \in \overline{k} \setminus k$. Then $\exp_C(\alpha\beta) \notin \overline{k}$.*

Previously he had relied more on the ingenious use of formal identities, à la Carlitz. After Wade's isolated early achievements, the real explosion of "early modern" transcendence results for function fields occurred through the work of G. Anderson, D. Thakur, and J. Yu (and many others).

2.5. **Yu's work.** In his first papers, Yu obtained the important analogues of both the Hermite, here the transcendence of non-zero period, and Gelfond-Schneider results for arbitrary Drinfeld modules defined over $\overline{k}$, i.e. with $\phi_\Lambda(\theta) \in \overline{K}\{F\}$ with K necessarily a finite extension of k.

Theorem 2.5.1 ([Yu 1986])**.** *Let ϕ_Λ be a Drinfeld module defined over $\overline{k}$.*

1) (à la Hermite) If λ is non-zero in Λ, then $\lambda \notin \overline{k}$.

2) (à la Gelfond-Schneider) If $u \neq 0$ with $\exp_\Lambda(u) \in \overline{k}$ and $\beta \notin \mathrm{End}(\lambda)$, then $\exp_\lambda(u\beta) \notin \overline{k}$.

2.5.2. *Ingredients in proofs.* Not only were the results motivated by the eponymous classical theorems, but so were the proofs. The ingredients were familiar:

(1) Construction of an auxiliary function which is a polynomial in the coordinate functions of the exponential map with lots of zeros (to force small values) on a lattice.
(2) Upper bound on the height of non-zero value at a "first" lattice point.
(3) Lower bound on this absolute value.
(4) Upper bound on this absolute value (Schwarz Lemma).

The most radical departure from the previous proofs was perhaps the application of the Riemann-Roch theorem in the place of (in the use of) the Box Principle in the Thue-Siegel construction of the auxiliary function.

2.5.3. *Analogue of Schneider's theorems.* Moreover Yu showed the analogues of Schneider's theorems for elliptic functions:

Theorem 2.5.4 ([Yu 1986]). *Let* $\exp_\lambda(z)$ *be defined over* $\overline{k}$ *with* $\operatorname{rank}_A \Lambda = 2$*, say* $\Lambda = A\ \lambda_1 + A\ \lambda_2$*. If* $\operatorname{End}\Lambda = A$*, then* λ_1/λ_2 *is transcendental.*

However to go further, it would be necessary to treat several Drinfeld modules simultaneously, i.e. to treat products of Drinfeld modules. For that, Anderson's t-modules are just what the doctor ordered. They are higher-dimensional generalizations of Drinfeld modules.

3. t-Modules

3.1. **Definitions.**

Drinfeld A-modules	Anderson t-modules
$\phi(\theta) \in \mathbb{C}_\infty\{F\}$	$\Phi(\theta) \in \operatorname{Mat}_n(\mathbb{C}_\infty)\{F\}$
$\phi(\theta) = \theta F^0 +$ higher order terms	$\Phi(\theta) = \theta M_0 +$ higher order terms
$d\phi = \theta$	$d\Phi = M_0 = \theta I_n + N \in \operatorname{Mat}_n(\mathbb{C}_\infty)$
empty condition	$N^n = 0$
unique entire $\exp_\lambda \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$	existence of unique entire $\exp_\Phi \colon \mathbb{C}_\infty^n \to \mathbb{C}_\infty^n$
$\exp_\Lambda(\theta z) = \phi(\theta)(\exp_\Lambda(z))$	$\exp_\Phi(M_0\mathbf{z}) = \Phi(\theta)\exp_\Phi(z)$
$\exp_\Lambda = F^0 +$ higher terms	$\exp_\Phi = I_n F^0 +$ higher terms with matrix coefficients

Just as for Drinfeld modules, the functional equation of the exponential function has a unique solution

3.1.1. *Zeros of exponential functions.* Just as in the case of classical abelian varieties, we cannot expect every t-module to be a torus, although some can be products of them as we shall see.

In this setting, a *sub-t-module* is an connected algebraic subgroup H of $\mathbb{G}_a^n$ such that

- for some $\mathbb{C}_\infty$-subspace $\operatorname{Lie}(H)$ of $\mathbb{C}_\infty^n$,
- $d\Phi(\theta)\operatorname{Lie}(H) \subseteq \operatorname{Lie}(H)$, and
- $\exp_\Phi(\operatorname{Lie} H) = H(\mathbb{C}_\infty)$,

In other words, t-modules satisfy the usual Lie correspondence for algebraic groups.

3.1.2. *Uniformizability of t-modules.* The question of the surjectivity of the exponential map is somewhat subtle, and we will not pursue it here, except to note that the exponential maps occurring here are surjective.

3.1.3. *Morphisms of t-modules.* Let $G = (\Phi, \mathbb{G}_a^r)$ and $Q = (\Psi, \mathbb{G}_a^s)$ denote two t-modules of respective dimensions r and s. Then by a *morphism* $f\colon Q \to G$, we mean a morphism of commutative algebraic groups $f\colon \mathbb{G}_a^s \to \mathbb{G}_a^r$ commuting with the action of A:

$$f\Psi(t) = \Phi(t)f.$$

3.2. **Yu's sub-t-module theorem.** In the setting of t-modules, Yu achieved the full analogue of the culmination of the Gelfond-Schneider setting in the following:

Theorem 3.2.1 (Yu's Theorem of the Invariant Vector Space). *Let $G = (\mathbb{G}_a^n, \Phi)$ be a t-module defined over $\overline{k}$. Let $\mathbf{u} \in \mathbb{C}_\infty^n$ (interpreted as $\mathrm{Lie}_G(\mathbb{C}_\infty)$ such that $\exp_\Phi(\mathbf{u}) \in G(\overline{k})$. Then the smallest vector space in $\mathrm{Lie}_G(\mathbb{C}_\infty)$ defined over $\overline{k}$ which is invariant under $d\Phi(\theta)$ and which contains $\mathbf{u}$ is the tangent space at the origin of a t-submodule of G.*

The proof of this is interesting, but it would lead us far afield into the area of effective elimination theory and applications to so-called zero estimates. Although students of mine have worked on the first topic, there is unfortunately no modern explication in the open literature of these facts of life over fields of characteristic zero.

3.3. **Yu's version of Baker's theorem.** To see how this applies, we consider the analogue of the celebrated theorem that garnered Baker a Fields Medal: In it, we let $\mathrm{End}_\Lambda := \{c \in \mathbb{C}_\infty : c\Lambda \subseteq \Lambda\}$.

Theorem 3.3.1 (Baker-Yu [Yu 1997]). *Let ϕ_Λ be defined over $\overline{k}$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}_\infty$ be linearly independent over End_Λ and assume that each $\exp_\Lambda(\alpha_i) \in \overline{k}$. Then*

$$1, \alpha_1, \ldots, \alpha_n$$

are linearly independent over $\overline{k}$.

3.4. **Proof of Baker-Yu.** Since Λ is fixed, let us set $\phi := \phi_\Lambda$ and $\exp := \exp_\Lambda$ for the moment.

3.4.1. *Construction of t-module.* We first have to define a t-module: Take $\Phi := \mathrm{diag}(\mathrm{id}, \phi, \ldots, \phi)$, where ϕ occurs n times on the diagonal.

Now check that Φ satisfies the necessary properties.

- $\Phi(\theta) \in \mathrm{Mat}_{n+1}(\overline{k})\{F\}$
- $d\Phi(\theta) = \theta I_{n+1}$, where I_{n+1} is the identity matrix in $\mathrm{Mat}_{n+1}(\overline{k})$, and N is the zero matrix.

From the functional equation for ϕ, we see that, if we provisionally set

$$\exp_\Phi(\mathbf{z}) := \mathrm{diag}(z_0, \exp(z_1), \ldots, \exp(z_n))$$

for the (column) vector of variables $\mathbf{z} := (z_0, z_1, \ldots, z_n)^t$, then $\exp_\Phi(\mathbf{z}) = I_n\mathbf{z} +$ higher terms, and we have the functional equation

$$\begin{aligned} \exp_\Phi(d\Phi(\theta)\mathbf{z}) = \exp_\Phi(\theta\mathbf{z}) &= \mathrm{diag}(\theta z_0, \exp(\theta z_1), \ldots, \exp(\theta z_n)) \\ &= \mathrm{diag}(\theta z_0, \phi(\theta)\exp(z_1), \ldots, \phi(\theta)\exp(z_n)) = \Phi(\theta)\exp_\Phi(\mathbf{z}). \end{aligned}$$

So with the above choice of Φ and $\exp_\Phi$ we have a t-module G of dimension n defined over $\overline{k}$.

3.4.2. *Specification of the point* $\mathbf{u}$. To apply Yu's Invariant Subspace Theorem, we need to specify a point $\mathbf{u}$ with $\exp_\Phi(\mathbf{u}) \in G(\overline{k})$. We simply take $\mathbf{u} := (1, \alpha_1, \ldots, \alpha_n)^t$ and find that indeed $\exp_\Phi(\mathbf{u}) \in \overline{k}^{n+1}$.

3.4.3. *Invariant subspaces.* Since $d\Phi(t\theta) = \mathrm{diag}(\theta, \ldots, \theta)$, multiplicaton of a vector by $d\Phi(\theta)$ is simply scalar multiplication by θ. As $\mathrm{Lie}\, H$ must be given by linear equations, the condition that $\mathrm{Lie}\, H$ should be closed under the action of $d\Phi(\theta)$ poses no additional restriction.

3.4.4. *Equations for* $\mathrm{Lie}\, H$*.* So only the questions remaining are:

(1) What linear k-subspaces V of $\mathrm{Lie}\, G$ have the property that for all $\mathbf{v} \in V$, all the values $\exp_\Phi(\mathbf{v})$ satisfy some fixed non-trivial polynomials over $\mathbb{C}_\infty$?
(2) Can our point above $\mathbf{u}$ lie in such a V?

Question (1): This question has an elementary, though not immediately apparent answer. From the functional equation for exp we know that, for every $\mathbf{z} \in \mathbb{C}_\infty^{n+1}$

$$\exp_\Phi(\mathbf{z}) = \mathbf{z} + M_1\mathbf{z}^{(1)} + M_2\mathbf{z}^{(2)} + \ldots.$$

From the corresponding $\mathbb{F}_q$-linearity of $\exp_\Phi(z)$, any minimal polynomial (say with respect to some lexicographical ordering of monomials) $P(x_0, \ldots, x_n)$ satisfied by w and the coordinates of $\exp_\Phi(w\mathbf{v})$ for all $w \in \mathbb{C}_\infty$ must itself have the special form $x_0 = 0$ or

$$P(\mathbf{x}) = \psi_1 x_1 + \psi_2 x_2 + \cdots + \psi_n x_n,$$

where the ψ_i, $i = 1, \ldots, n$ are $\mathbb{F}_q$-linear. Using that, we can apply the functional equation, treating w as the variable that it is, to find that, in fact

(a) x_0 does not occur in P and
(b) the coordinates of $\mathbf{v}$ must be linearly dependent over $\mathrm{End}(\phi) \subset \mathbb{C}_\infty$.

Question (2): According to the previous sentence, the hypotheses on $\mathbf{u}$ do not allow it to lie in $\mathrm{Lie}\, H$ for any sub-t-module of Φ. The assertion is proven.

This situation is the analogue of a result of Kolchin for commutative algebraic groups:

Theorem 3.4.5 ([Kolchin 1968])**.** *Algebraic subgroups of products of simple algebraic groups are determined in the tangent space by linear relations corresponding to isogenies.*

The corresponding result holds for products of Drinfeld modules. See [Yu 1997].

3.5. **Quasi-periodic functions.** If one is led by Drinfeld to natural function field analogies with elliptic curves, one can also be inspired to pursue a further analogy with the elliptic situation – that of extensions of elliptic curves $\mathcal{E}$ by the additive group $\mathbb{G}_a$, giving rise to quasi-elliptic functions:

$$0 \to \mathbb{G}_a \to E \to \mathcal{E} \to 0$$

where the elliptic function of E is given by

$$(z, w) \longmapsto (1, \wp(z), \wp'(z), w - \zeta(z))$$

and $\zeta(z)$ is the quasi-periodic Weierstraß zeta function. The periods of this map are the pairs (ω, η), where $\omega = n_1\omega_1 + n_2\omega_2$ is a period of $\wp(z)$ and $\eta = n_1\eta_1 + n_2\eta_2$ is the corresponding quasi-period expressed in terms of a basis ω_1, ω_2 of periods for $\wp(z)$ and $n_i = 2\zeta(\omega_i/2)$, $i = 1, 2$. Deligne, Anderson, and Yu developed a theory of quasi-periodic Drinfeld functions, which was recast by Gekeler. See [Gekeler] and [Yu 1990].

3.5.1. *Bi-derivations.* A *ϕ-biderivation* is an $\mathbb{F}_q$-linear map $\delta\colon A \to \mathbb{C}_\infty\{F\}F$ satisfying

$$\delta(ab) = a\delta(b) + \delta(a)\phi(b),$$

for all $a, b \in A$.

The space $D(\phi)$ of ϕ-biderivations splits as a direct sum

$$D(\phi) = D_{sr}(\phi) \oplus \mathbb{C}_\infty\delta_0 \oplus D_{si}(\phi),$$

where

- $D_{sr}(\phi) = \{\delta \in D(\phi)\colon \deg_F \delta(a) < \deg_F \phi(a), \forall a \in A\}$,
- $\delta_0(a) = \phi(a) - aF^0$,
- $D_{si}(\phi) = \{\delta \in D(\phi)\colon \deg_F \delta(a) > \deg_F \phi(a), \forall \text{ non-constant } a \in A\}$.

3.5.2. *Quasi-periodic functions.* For each ϕ-biderivation δ, there is a unique entire $\mathbb{F}_q$-linear function $F_\delta(z)$, with no linear term, such that for every $a \in A$

$$F(az) - aF(z) = \delta(a)\exp(z).$$

It is a fact that if $r = \operatorname{rank}_A \Lambda$, then $\dim_{\mathbb{C}_\infty} D_{sr}(\phi) = r - 1$ (see [Gekeler]). If $\phi(\theta) \in K\{F\}$, $K \subset \overline{k}$, then there is a basis $\delta_1, \dots, \delta_{r-1}$ such that each $\delta_i(\theta) \in K\{F\}F$. Then each corresponding quasi-periodic $F_i(z)$ has coefficients from K.

It is straight-forward to verify that $F_{\delta_0}(z) = \exp(z)$, and it is not hard to verify that for $\delta \in D_{si}(\phi)$, the corresponding $F(z)$ is expressible algebraically in terms of $z, \exp(z), F_1(z), \dots, F_{r-1}(z)$. So to obtain maximal algebraic independence, we can restrict our attention to these functions.

3.5.3. *Quasi-periodic t-module.* So now we are ready to define our quasi-periodic t-module

$$\Phi(\theta) = \begin{pmatrix} \phi(\theta) & 0 & 0 & \dots & 0 \\ \delta_1(\theta) & \theta F^0 & 0 & \dots & 0 \\ \delta_2(\theta) & 0 & \theta F^0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \delta_{r-1}(\theta) & 0 & \dots & 0 & \theta F^0 \end{pmatrix}$$

The corresponding exponential function is

$$\exp_\Phi\colon \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{r-1} \end{pmatrix} \longmapsto \begin{pmatrix} \exp(z_0) \\ z_1 + F_1(z_0) \\ \vdots \\ z_{r-1} + F_{r-1}(z_0) \end{pmatrix}$$

3.5.4. *Quasi-periodic Baker-Yu theorem.* As a consequence of Yu's Invariant Vector Space Theorem, we can extend Baker-Yu to the following:

Theorem 3.5.5 (see [Brownawell 1998], [Brownawell-Papanikolas 2002])**.** *Let $\phi = \phi_\Lambda$ be defined over $K \subset \overline{k}$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}_\infty$ be linearly independent over* $\operatorname{End}_\Lambda$ *and assume that each* $\exp_\Lambda(\alpha_i) \in \overline{k}$. *If* $\operatorname{rank}_A \Lambda = r$, *let $F_1(z), \dots, F_{r-1}(z)$ be the quasi-periodic functions corresponding to $r-1$ K-linearly independent strictly reduced ϕ-biderivations.*

Then the $1 + rn$ *numbers*

$$1, \alpha_1, \ldots, \alpha_n; F_1(\alpha_1), \ldots, F_1(\alpha_n); \ldots, F_{r-1}(\alpha_n)$$

are linearly independent over $\overline{k}$*.*

3.5.6. *Conjectured independence for quasi-periodic* t*-modules.* In fact, I have been so bold as to make the following conjecture:

Conjecture 3.5.7 (Schanuel-Like conjecture for Drinfeld Modules)**.** *For* $i = 1, \ldots, n$*, let* Λ_i *be non-isogenous* A*-lattice of rank* r_i *over the multiplications* $\mathcal{O}_i$ *of* Λ_i *and let the set* $U_i \subset \mathbb{C}_\infty$ *be* $\mathcal{O}_i$*-linearly independent. Let* $\exp_i(z)$ *denote the corresponding Drinfeld exponential functions and* $F_{ij}(z)$*,* $j = 2, \ldots, r_i$*, the quasi-periodic functions defined above. Then among the following* $\sum_{i=1}^{n}(1 + r_i)|U_i|$ *numbers:*

$$u, \quad \exp_i(u), \quad F_{ij}(u), \quad j = 2, \ldots, r_i,$$

$u \in U_i$*,* $i = 1, \ldots, n$*, at least* $\sum_i r_i|U_i|$ *are algebraically independent.*

There is as yet very little evidence for this conjecture. But Mark Twain was once asked: “Why go out on a limb if you don’t have to?” He replied: “Because that’s where the fruit is.”

The results presented in this first part were stimulated in large part by the close analogy between the situation in the classical number theoretical situation and function fields. However they ignore one essential fact: θ is transcendental over the prime field. In particular, it means that we have a natural underlying operator that does not exist in the classical setting: differentiation.

3.5.8. *Biderivations for* t*-modules.* Before we do that however, we note that the notion of biderivation extends naturally to the setting of a t-module Φ:

A Φ-*biderivation* is an $\mathbb{F}_q$-linear map $\boldsymbol{\delta} : A \to F\,\mathrm{Mat}_{1\times r}(\mathbb{C}_\infty)\{F\}$ satisfying the product formula that, for all a, $b \in A$,

$$\boldsymbol{\delta}(ab) = a\boldsymbol{\delta}(b) + \boldsymbol{\delta}(a)\Phi(b).$$

Set

$$N^\perp := N^\perp(\mathbb{C}_\infty) := \{V \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty F^0) : VN = 0\} \subset \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty F^0),$$

where N is the nilpotent part of $d\Phi(\theta) = \theta I_r + N$.

Let $\boldsymbol{U} = (U_1, \ldots, U_r) \in (\mathbb{C}_\infty\{F\})^r$ with $d\boldsymbol{U} \in N^\perp(\mathbb{C}_\infty)$, where $d\boldsymbol{U} = (dU_1, \ldots, dU_r)$ denotes the vector of coefficients of F^0 in $\boldsymbol{U}$. We define $\boldsymbol{\delta}^{(\boldsymbol{U})} : A \to M(E)$ via

$$\boldsymbol{\delta}^{(\boldsymbol{U})}(a) := \boldsymbol{U}\Phi(a) - a\boldsymbol{U}, \tag{3.5.8.1}$$

for every $a \in A$. The condition $d\boldsymbol{U} \in N^\perp$ is equivalent to saying that $d\boldsymbol{\delta}^{(\boldsymbol{U})}(\theta)N = 0$, i.e. that $\boldsymbol{\delta}^{(\boldsymbol{U})}(\theta) \in F\,\mathrm{Mat}_{1\times r}(\mathbb{C}_\infty)\{F\}$. The maps $\boldsymbol{\delta}^{(\boldsymbol{U})}$ are A-biderivations, and we call them *inner*. When $\boldsymbol{U} \in F\,\mathrm{Mat}_{1\times r}(\mathbb{C}_\infty)\{F\}$, we call $\boldsymbol{\delta}^{(\boldsymbol{U})}$ *strictly inner*; we denote the $\mathbb{C}_\infty$-vector space of strictly inner biderivations by $\mathrm{Der}(\Phi)_{si}$ and set $H_{\mathrm{DR}}(\Phi, \mathbb{C}_\infty) := \mathrm{Der}(\Phi)/\mathrm{Der}_{si}(\Phi)$ and set $H(\Phi, \mathbb{C}_\infty) := \mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty)$.

To each biderivation $\boldsymbol{\delta}$, there is a unique associated quasi-periodic function $F_{\boldsymbol{\delta}}$ satisfying:

$$F_{\boldsymbol{\delta}}(d\Phi(a)\mathbf{z}) = aF_{\boldsymbol{\delta}}(\mathbf{z}) + \boldsymbol{\delta}(a)\exp_{\Phi}(\mathbf{z}), \tag{3.5.8.2}$$

$$F_{\boldsymbol{\delta}}(\mathbf{z}) \equiv 0 \pmod{\mathbf{z}^q}, \tag{3.5.8.3}$$

where the latter condition means that every non-zero term in the power series $F_{\boldsymbol{\delta}}(\mathbf{z})$ is that of a monomial $z_i^{q^h}$ with $h > 0$.

3.5.9. *Extensions of t-modules arising from biderivations.* Let $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_s$ be Φ-biderivations which are linearly independent modulo the inner Φ-biderivations. Then the t-module

$$\theta \mapsto \begin{pmatrix} \Phi(\theta) & 0 & 0 & \ldots & 0 & \\ \boldsymbol{\delta}_1(\theta) & \theta I_r & 0 & \ldots & 0 & \\ \vdots & & & & \vdots & \\ \boldsymbol{\delta}_s(\theta) & 0 & & & \ldots & \theta I_r \end{pmatrix}$$

has as exponential function

$$\begin{pmatrix} \mathbf{z} \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_s \end{pmatrix} \mapsto \begin{pmatrix} \exp_{\Phi}(\mathbf{z}) \\ \mathbf{z} + F_{\boldsymbol{\delta}_1}(\mathbf{w}_1) \\ \vdots \\ \mathbf{z} + F_{\boldsymbol{\delta}_s}(\mathbf{w}_s) \end{pmatrix}$$

Out of desperation, we denote this t-module as $\Phi_{\rtimes \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_s}$. Note that projection onto the first r factors gives a short exactsequence of t-modules

$$(\mathbb{G}_a^r)^s \to \Phi_{\rtimes \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_s} \to \Phi \to 0.$$

This is our first example of minimal extensions of t-modules. But it is not our only example. The other main one so far arises from taking derivatives.

3.6. **Derivatives and linear independence.** L. Denis [Denis 2000] first began the investigation of the independence of values of derivatives of logarithms of algebraic quantities. We need some background. Rather than than starting with the most general result, I want to consider the case furnished by divided derivatives.

3.6.1. *Divided derivatives.* Divided derivatives $D^{[i]}$, $i \geq 0$ are defined on A by setting

$$D^{[i]}\theta^n := (\theta^n)^{[i]} := \begin{cases} \binom{n}{i}\theta^{n-i} & n \leq i \\ 0 & n > i. \end{cases}$$

This extends uniquely, first to k and then to k^{sep} through the the product formula:

$$D^{[i]}(ab) = \sum_{j+l=i} D^{[j]}(a)D^{[l]}(b).$$

The main obvious advantage of divided derivatives over ordinary derivatives in characteristic p is that, while the iterated p-fold derivative vanishes identically and already $d/d\theta(\theta^p) = 0$, $D^{[p]}p(\theta^p) = 1$.

3.6.2. *Derived t-modules.* Let Φ be a d-dimensional t-module defined over $\overline{k}^{sep}$ and let $\exp_\Phi(\mathbf{u}) \in \mathbb{G}_a(\overline{k})^d$. For $s \in \mathbb{Z}_{\geq 0}$, consider the $(s+1)d \times (s+1)d$ matrix $\Phi^{[s]}(\theta)$ consisting of $d \times d$ blocks

$$\Phi_{ij}(\theta) = \sum_{j=0}^{m} A_m^{[i-q^m j]} F^m,$$

where $\Phi(\theta) = \sum_{j=0}^m A_m F^m$, $A_m \in \mathrm{Mat}_{d\times d}(\overline{k}^{sep})$:

$$\Phi^{[s]}(t) = \left(P_{ij}(\theta)\right), \qquad \Phi^{[0]}(t) = \Phi(t)$$

Then it is straightforward, but not trivial, to verify that, for $\mathbf{u} \in \overline{k}^{sep}$

$$\exp_{\Phi^{[s]}}(\mathbf{u}, \mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[s]}) = (\exp_\Phi(\mathbf{u}), \exp_\Phi(\mathbf{u})^{[1]}, \ldots, \exp_\Phi(\mathbf{u})^{[s]}).$$

3.7. **Minimal extensions of t-modules by $\mathbb{G}_a$.** To treat both the extensions of t-modules obtained through biderivations and through divided derivatives (and of course any mixture of them) at once, we introduce the notion of minimal extensions of t-modules. We say that the surjective morphism of t-modules

$$Q \xrightarrow{\pi} G \to 0$$

makes Q into a *minimal extension* or *Frattini cover* of G if no proper t-submodule of Q maps onto G.

Quasi-periodic t-modules and derived t-modules given above are interesting examples of minimal extensions of the underlying t-modules.

3.7.1. *Linear independence in minimal extensions.*

Theorem 3.7.2 ([Brownawell 2002])**.** *Let*

$$Q \xrightarrow{\pi} G \to 0$$

be a minimal extension of the t-module G, with Q, G, and π defined over $\overline{k}$. Let $\mathbf{u} \in \mathrm{Lie}\, Q$ with $\exp_G(\mathbf{u}) \in Q(\overline{k})$ and set $\mathbf{v} := d\pi(\mathbf{u})$.

Then if the coordinates of $\mathbf{v}$ are $\overline{k}[d\Phi(\theta)]$-linearly independent, so are the coordinates of $\mathbf{u}$.

3.7.3. *Linear independence of divided derivatives of logarithms of Drinfeld modules [Denis-Brownawell].* See [Brownawell 2002]. As a special case, consider $G := (\phi, \mathbb{G}_a)$, a Drinfeld module defined over $\overline{k}^{sep}$ and $\mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{C}_\infty^n$ with each $\exp_\phi(u_i) \in \overline{k}^{sep}$. If $u_1, \ldots, u_n$ are linearly independent over $\mathrm{End}(\phi)$, then

$$1; u_1, \ldots, u_n; \ldots; u_1^{[j]}, \ldots, u_n^{[j]}; \ldots$$

are $\overline{k}$-linearly independent.

3.7.4. *Concluding thoughts.* So far we have seen the following historical development:

- Drinfeld modules as analogues of elliptic curves.
- A shift in point of view from an analogy with elliptic curves in terms of exponential functions and lattice of zeros to the action of A as represented by $\phi(\theta)$.
- Generalization to higher dimension via the action of $\Phi(\theta)$ on $\mathbb{G}_a^d$.

The overall approach used the Dirichlet Box Principle in the guise of the Thue-Siegel Lemma and the functional equation on the exponential *function* whose values we are considering.

However the original attitude by Carlitz and his student Wade that the formal functional, almost combinatorial, properties of the elements viewed as power series should guarantee that they satisfy no surprising formal relations, even if we are ignorant of any classical analogues of the expressions we are considering.

4. The "ABP-criterion"

4.1. **Algebraic curves and functions.** In terms of transcendence in positive characteristic, the quantities that interest us are contained in the field $\mathbb{C}_\infty$. However, as is often the case, these quantities also arise as values of special functions on $\mathbb{C}_\infty$, and we have already seen this with values and zeros of the Carlitz exponential function. For the most part, all functions we will consider will be elements of the Laurent series field $\mathbb{C}_\infty((t))$, but we will start at the beginning.

4.1.1. *Rational functions.* Here we will be a little more systematic about what kinds of functions will (or may) produce quantities of particular interest to us. We begin with the curve $\mathbb{P}^1/\mathbb{F}_q$ and its function field $\mathbb{F}_q(t)$,

$$\mathbb{P}^1/\mathbb{F}_q \longleftrightarrow \mathbb{F}_q(t).$$

For any field $K \supseteq \mathbb{F}_q$, we can extend scalars to K and consider rational functions on $\mathbb{P}^1$ over K,

$$\mathbb{P}^1/K \longleftrightarrow K(t).$$

In particular we will often take $K = \overline{k}$ or $K = \mathbb{C}_\infty$.

4.1.2. *Analytic functions.* We first define the *Tate algebra* of functions that are analytic on the closed unit disk in $\mathbb{C}_\infty$:

$$\mathbb{T} := \left\{ \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |a_i|_\infty \to 0 \right\}.$$

The Tate algebra $\mathbb{T}$ is a principal ideal domain with maximal ideals generated by $t-a$ for all $a \in \mathbb{C}_\infty$, $|a|_\infty \leq 1$. Moreover, each non-zero function $f \in \mathbb{T}$ has a unique factorization of the form

$$f = \lambda \left[\prod_{|a|_\infty \leq 1} (t-a)^{\mathrm{ord}_a(f)} \right] \left[1 + \sum_{i=1}^{\infty} b_i t^i \right],$$

where $\lambda \in \mathbb{C}_\infty$, $\sup |b_i|_\infty < 1$, and $|b_i|_\infty \to 0$. See Section 2.2 of [Fresnel-van der Put] for more details. Of some importance to us is the fact that

$$\mathbb{T} \cap \mathbb{F}_q[[t]] = \mathbb{F}_q[t]. \tag{4.1.2.1}$$

We set $\mathbb{L}$ to be the fraction field of $\mathbb{T}$, which is a subfield of the Laurent series field $\mathbb{C}_\infty((t))$.

4.1.3. *Entire functions.* The ring $\mathbb{E}$ of *entire functions* is defined to be

$$\mathbb{E} := \left\{ \sum_{i \geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; \begin{array}{c} \sqrt[i]{|a_i|_\infty} \to 0, \\ [k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty \end{array} \right\}.$$

The first condition implies that a given $f \in \mathbb{E}$ converges on all of $\mathbb{C}_\infty$. It is equivalent to having

$$\lim_{i \to \infty} \frac{1}{i} \operatorname{ord}_\infty(a_i) = \infty.$$

The second condition implies that $f(\overline{k_\infty}) \subseteq \overline{k_\infty}$.

4.1.4. *Frobenius twisting.* Let $f = \sum a_i t^i \in \mathbb{C}_\infty((t))$. For any $n \in \mathbb{Z}$, we set

$$f^{(n)} := \sum a_i^{q^n} t^i \in \mathbb{C}_\infty((t)).$$

Thus $f \mapsto f^{(n)}$ has the effect of simply raising the coefficients of f to the q^n-th power. The maps

$$f \mapsto f^{(n)} : \mathbb{C}_\infty((t)) \xrightarrow{\sim} \mathbb{C}_\infty((t))$$

are automorphisms. They also induce automorphisms of each of the following rings and fields:

$$\overline{k}[t], \quad \mathbb{T}, \quad \overline{k}(t), \quad \mathbb{L}, \quad \mathbb{E}.$$

When $n = -1$, we call this automorphism σ: i.e., $\sigma(f) = f^{(-1)}$. Moreover, σ has the following fixed rings and fields:

$$\mathbb{C}_\infty((t))^\sigma = \mathbb{F}_q((t)), \quad \overline{k}(t)^\sigma = \mathbb{F}_q(t), \quad \mathbb{T}^\sigma = \mathbb{F}_q[t], \quad \mathbb{L}^\sigma = \mathbb{F}_q(t).$$

As a matter of notation, if $A \in \operatorname{Mat}_{r \times s}(\mathbb{C}_\infty((t)))$, then we set $A^{(n)}$ to be the matrix with entries

$$\left(A^{(n)}\right)_{ij} = A_{ij}^{(n)}.$$

4.2. **"The two t's".**

As we can see from the definitions above, we have introduced a second variable "t" in addition to the variable we already had from the previous section "θ." In many respects these two variables look and act the same, and people are often confused that there is a need for both of them. Indeed in the first section of these notes, we tried to avoid these complications. See Ch. 5 of [Goss] for more explanation.

4.2.1. *Scalar quantities and operators.* For the remainder of these notes we will adhere to the following dichotomy:

- Scalars: Numbers built up out of θ; elements of $\mathbb{C}_\infty$.
- Functions and Operators: Functions built up out of t; elements of $\mathbb{C}_\infty((t))$.

One reason to do this is that most of the values that we consider arise as values of special functions which are defined on $\mathbb{C}_\infty$ or a subset of it. Another reason to do this is that, in the case of t-modules for example, we want to consider certain rings of functions as acting as operators on $\mathbb{C}_\infty$ or spaces builted out of it. As we saw in the previous section, the action of t on the Lie algebra of a t-module does not always coincide with scalar multiplication by

θ. It is thus useful to differentiate between the two. More differences between "t" and "θ" will become apparent in the rest of the notes.

4.2.2. *The Carlitz $\mathbb{F}_q[t]$-module.* As alluded to in the previous paragraph it is often useful to think of the Carlitz module as an $\mathbb{F}_q[t]$-module. More concretely, the Carlitz module C is an $\mathbb{F}_q[t]$-module, whose underlying additive group is $\mathbb{C}_\infty$. The operation of $\mathbb{F}_q[t]$ on C is given by the $\mathbb{F}_q$-algebra homomorphism,

$$C : \mathbb{F}_q[t] \to \mathbb{C}_\infty\{F\},$$

where

$$C(t) = \theta + F.$$

Thus the t-torsion on C is the set

$$\begin{aligned} C[t] &:= \{x \in \mathbb{C}_\infty \mid C(t)(x) = 0\} \\ &= \{x \in \mathbb{C}_\infty \mid \theta x + x^q = 0\}. \end{aligned}$$

Moreover, we see that

$$C[t] = \mathbb{F}_q\, \zeta_\theta,$$

where ζ_θ is a fixed choice of $(q-1)$-st root of $-\theta$. More generally for any $f(t) \in \mathbb{F}_q[t]$, the f-torsion on C is

$$C[f] = \{x \in \mathbb{C}_\infty \mid C(f)(x) = 0\} \cong \mathbb{F}_q[t]/(f),$$

where the isomorphism on the right is an isomorphism of $\mathbb{F}_q[t]$-modules. We can also check that

$$C[f] = \{\exp_C(\pi_q a/f(\theta)) \mid a \in \mathbb{F}_q[\theta]\}.$$

In this respect we can see that $\zeta_\theta = \exp_C(\pi_q/\theta)$.

One can (and we will!) adopt this convention for Drinfeld modules and t-modules in what follows—that they are $\mathbb{F}_q[t]$-modules in the natural way and that their endomorphism rings are extensions of $\mathbb{F}_q[t]$.

4.3. **The function $\Omega(t)$.** Some references for this section are §5.1 of [Anderson-Brownawell-Papanikolas] and §3.3 of [Papanikolas 2008].

4.3.1. *Definition.* Let $\zeta_\theta = \sqrt[q-1]{-\theta} = \exp_C(\pi_q/\theta)$ be a fixed choice of $(q-1)$-st root of $-\theta$ as above. We define an infinite product,

$$\Omega(t) := \zeta_\theta^{-q} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in k_\infty(\zeta_\theta)[[t]] \subseteq \mathbb{C}_\infty((t)).$$

4.3.2. $\Omega(t)$ *is entire.* If we expand $\Omega(t)$, we see that

$$\zeta_\theta^q \Omega(t) = \sum_{i=1}^{\infty} a_i t^i = 1 - \sum_{j \geq 1} \frac{t}{\theta^{q^j}} + \sum_{j_2 \geq j_1 \geq 1} \frac{t^2}{\theta^{q^{j_1}} \theta^{q^{j_2}}} + \cdots .$$

It is not hard to see that

$$\operatorname{ord}_\infty(a_1) = q$$
$$\operatorname{ord}_\infty(a_2) = q + q^2,$$

and in general,

$$\operatorname{ord}_\infty(a_i) = q + q^2 + \cdots + q^i \geq q^i.$$

Clearly then $\frac{1}{i} \operatorname{ord}(a_i) \to \infty$ and $\Omega(t)$ converges on all of $\mathbb{C}_\infty$. That all of the coefficients a_i lie in a finite extension of k_∞ follows straight from the definition, and so $\Omega(t) \in \mathbb{E}$.

4.3.3. $\Omega(t) \in \mathbb{T}^\times$. The zeros of $\Omega(t)$ on $\mathbb{C}_\infty$ are precisely

$$t = \theta^q, \theta^{q^2}, \theta^{q^3}, \ldots,$$

all of which have ∞-adic absolute value strictly bigger than 1. Thus $1/\Omega(t)$ is well defined on the closed unit disk in $\mathbb{C}_\infty$ and so is in $\mathbb{T}$. In fact, $1/\Omega(t)$ converges for all $t = \alpha$ with $|\alpha|_\infty < |\theta^q|_\infty$.

4.3.4. *Connection with the Carlitz period.* If we recall the formula for the period π_q of the Carlitz module for $\mathbb{F}_q[t]$, as defined in §2.1.1, it follows that π_q is related to the specialization $\Omega(\theta)$. Specifically,

$$\boxed{\pi_q = -\frac{1}{\Omega(\theta)}.} \tag{4.3.4.1}$$

4.3.5. *Functional equation for $\Omega(t)$.* Let us define

$$\Omega^{(-1)}(t) := \zeta_\theta^{-1} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^{i-1}}}\right),$$

which is obtained by taking the q-th root of all of the coefficients of the power series $\Omega(t)$. We claim that

$$\boxed{\Omega^{(-1)}(t) = (t - \theta)\Omega(t).} \tag{4.3.5.1}$$

To see this, it is evident that

$$\Omega^{(-1)}(t) = \zeta_\theta^{-1} \left(1 - \frac{t}{\theta}\right) \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right).$$

Thus to prove (4.3.5.1) it remains to show that

$$-\theta \zeta_\theta = \zeta_\theta^q,$$

and this follows from the fact that $\zeta_\theta = \sqrt[q-1]{-\theta}$.

4.3.6. *Power series expansion of* $1/\Omega^{(-1)}(t)$. The following formula shows that $1/\Omega^{(-1)}(t)$ is a generating function for division values of the Carlitz exponential. See Prop. 5.1.3 of [Anderson-Brownawell-Papanikolas]. We claim that

$$\frac{1}{\Omega^{(-1)}(t)} = \sum_{i=0}^{\infty} \exp_C\left(\pi_q/\theta^{i+1}\right)t^i.$$

To see this, first of all both sides are elements of $\mathbb{T}^\times$ (check!) and both have the same constant term ζ_θ since $\exp_C(\pi_q/\theta) = \zeta_\theta$. Now set

$$H(t) = \sum_{i=0}^{\infty} \exp_C\left(\pi_q/\theta^{i+1}\right)^q t^i.$$

Our claim is that $H^{(-1)} = 1/\Omega^{(-1)}$. The functional equation (2.4.1.1) for the Carlitz exponential says that

$$\exp_C(\pi_q/\theta^i) = \theta\exp_C(\pi_q/\theta^{i+1}) + \exp_C(\pi_q/\theta^{i+1})^q,$$

and so we see that

$$\begin{aligned} H(t) &= \sum_{i=0}^{\infty}\left(\exp_C(\pi_q/\theta^i) - \theta\exp_C(\pi_q/\theta^{i+1})\right)t^i, \\ &= tH^{(-1)}(t) - \theta H^{(-1)}(t), \\ &= (t-\theta)H^{(-1)}(t). \end{aligned}$$

It follows that

$$(H\Omega)^{(-1)} = H\Omega,$$

and so $H\Omega \in \mathbb{T}^\sigma = \mathbb{F}_q[t]$. After slightly more work we can check that $H\Omega = 1$.

4.4. **The ABP-criterion.** The following theorem is the fundamental basis for all of the algebraic independence theorems we will consider. In some sense, it works as a t-motivic analogue of Yu's Sub-t-Module Theorem (Theorem 3.2.1), which is a statement we will explain more fully in later sections.

Theorem 4.4.1 ([Anderson-Brownawell-Papanikolas], Thm. 3.1.1)**.** *Let* $r \geq 1$. *Fix a matrix*

$$\Phi = \Phi(t) \in \mathrm{Mat}_{r\times r}(\overline{k}[t]),$$

such that $\det(\Phi) = c(t-\theta)^s$ *for some* $c \in \overline{k}^\times$ *and* $s \geq 0$. *Suppose we have also a column vector of entire functions,*

$$\psi = \psi(t) \in \mathrm{Mat}_{r\times 1}(\mathbb{E}),$$

satisfying the functional equations

$$\psi^{(-1)} = \Phi\psi.$$

Now suppose that there is a $\overline{k}$*-linear relation among the entries of* $\psi(\theta)$*; that is, there is a row vector* $\rho \in \mathrm{Mat}_{1\times r}(\overline{k})$ *so that*

$$\rho\psi(\theta) = 0.$$

Then there is a row vector of polynomials $P(t) \in \mathrm{Mat}_{1\times r}(\overline{k}[t])$ *so that*

$$P(t)\psi(t) = 0, \quad P(\theta) = \rho.$$

4.4.2. *Proof of Theorem 4.4.1.* Our emphasis will mostly be on applications of this theorem, but a few words about its proof are in order. We first consider the case when $r = 1$. Then Φ and ψ are really functions, and

$$\rho\psi(\theta) = 0.$$

Without loss of generality we can assume $\rho \neq 0$, and so $\psi(\theta) = 0$. Therefore, as we are looking for a non-zero polynomial P so that $P\psi = 0$, we need to show that ψ is identically zero. Consider for $n \geq 0$:

$$\psi\bigl(\theta^{q^{-n}}\bigr)^{q^{-1}} = \psi^{(-1)}\bigl(\theta^{q^{-(n+1)}}\bigr) = \Phi\bigl(\theta^{q^{-(n+1)}}\bigr)\psi\bigl(\theta^{q^{-(n+1)}}\bigr),$$
$$\Phi\bigl(\theta^{q^{-(n+1)}}\bigr) \neq 0 \quad \text{(by hypothesis)}.$$

By induction, it follows that for all $n \geq 0$ we have $\psi\bigl(\theta^{q^{-n}}\bigr) = 0$. But then ψ has infinitely many zeros in the closed unit disk of $\mathbb{C}_\infty$, and so it must be identically zero (via the factorization in §4.1.2.

When $r \geq 2$, the proof is necessarily more complicated. We consider functions $E \in \mathbb{E}$ of the form,

$$E(t) = P(t)\psi(t), \quad P(t) \in \mathrm{Mat}_{r\times 1}(\overline{k}[t]).$$

Now the field $\overline{k}$ has transcendence degree 1 over $\mathbb{F}_q$, and so finitely generated subfields of $\overline{k}$ are themselves function fields of curves over finite fields. This allows us to use the Riemann-Roch theorem to prove various box principle estimates in the vein of Siegel and Thue. These estimates prove the existence of an auxiliary function E that vanishes identically once we consider a competing Liouville estimate. By controlling the degree of $P(t)$ and the size of its coefficients, we can show that a particular choice P will also satisfy the condition that $P(\theta) = \rho$.

4.4.3. *Proof of Wade's theorem.* Here we will show how to use Theorem 4.4.1 to prove that π_q is transcendental. It of course suffices to show that $-\frac{1}{\pi_q} = \Omega(\theta)$ is transcendental. Suppose we have a polynomial relation,

$$\sum_{i=0}^{m} \rho_i \Omega(\theta)^i = 0, \quad \rho_i \in \overline{k},\ \rho_0\rho_m \neq 0.$$

Now let $\Phi \in \mathrm{Mat}_{m+1}(\overline{k}[t])$ be the diagonal matrix,

$$\Phi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & t-\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (t-\theta)^m \end{bmatrix},$$

and let $\psi \in \mathrm{Mat}_{(m+1)\times 1}(\mathbb{E})$ be defined by

$$\psi = \begin{bmatrix} 1 \\ \Omega(t) \\ \vdots \\ \Omega(t)^m \end{bmatrix}.$$

By the functional equation of Ω, we see that

$$\psi^{(-1)} = \Phi\psi.$$

Theorem 4.4.1 then provides the existence of polynomials $P_i(t) \in \overline{k}[t]$ so that

$$\sum_{i=0}^{m} P_i(t)\Omega(t)^i = 0, \quad P_i(\theta) = \rho_i.$$

Since $\rho_m \neq 0$, it follows that $P_0(t)$ must vanish at all of the zeros of $\Omega(t)$. Since a polynomial cannot have infinitely many zeros, we must have $P_0(t) = 0$ identically, which contradicts that $P_0(\theta) = \rho_0 \neq 0$.

5. Difference equations

In this section we investigate some examples of matrices Φ and ψ to which one may potentially apply Theorem 4.4.1. Indeed many of the examples in this section will appear out of the blue, but it will be useful to have these examples in mind as we proceed. Their origins will be made clear in §6 and we will revisit them in §8.

5.1. **Difference equations and their solutions.**

5.1.1. *Definitions.* Generally speaking, difference equations arise in the following setting. We have two fields $K \subseteq L$ together with an automorphism $\sigma : L \xrightarrow{\sim} L$ such that $\sigma : K \xrightarrow{\sim} K$. We further require that

$$L^\sigma = K^\sigma =: E.$$

Now given a matrix $\Phi \in \mathrm{GL}_r(K)$, we can consider the system of equations,

$$\sigma(\psi) = \Phi\psi, \qquad \psi \in \mathrm{Mat}_{r\times 1}(L).$$

Such a system is called a system of *difference equations.* See [van der Put-Singer] for more information. In the situation for Drinfeld modules, we take

$$L = \mathbb{L}, \quad K = \overline{k}(t), \quad E = \mathbb{F}_q(t).$$

5.1.2. *Solution spaces.* Now fix a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$. We do not yet require Φ to have polynomial entries. We consider the system of difference equations,

$$\psi^{(-1)} = \Phi\psi, \qquad \big(\sigma(\psi) = \Phi\psi\big), \tag{5.1.2.1}$$

where we look for solutions $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{L})$. Define the space

$$\mathrm{Sol}(\Phi) = \{\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{L}) \mid \psi^{(-1)} = \Phi\psi\}.$$

Now because of (5.1.2.1), it is clear that $\mathbb{L}$ is a vector space over $\mathbb{F}_q(t) = \mathbb{L}^\sigma$. In much the same way as one bounds dimensions of spaces of solutions of homogeneous differential equations, we find here that

$$\dim_{\mathbb{F}_q(t)} \mathrm{Sol}(\Phi) \leq r. \tag{5.1.2.2}$$

To verify (5.1.2.2), we will show that if $\psi_1, \ldots, \psi_m \in \mathrm{Sol}(\Phi)$ are linearly independent over $\mathbb{F}_q(t)$, then they are linearly independent over $\mathbb{L}$.

We proceed by contradiction. Suppose that $m \geq 2$ is minimal so that $\psi_1, \dots, \psi_m$ are linearly independent over $\mathbb{F}_q(t)$ but that

$$0 = \sum_{i=1}^{m} f_i \psi_i, \quad f_i \in \mathbb{L}, f_1 = 1.$$

Multiplying both sides on the left by Φ, we have

$$0 = \sum_{i=1}^{m} f_i \Phi \psi_i = \sum_{i=1}^{m} f_i \psi^{(-1)},$$

the second equality following from (5.1.2.1). Thus, subtracting the two equations we find,

$$0 = \sum_{i=1}^{m} (f_i - f_i^{(1)}) \psi_i = \sum_{i=2}^{m} (f_i - f_i^{(1)}) \psi_i,$$

where the second equality follows from the choice of $f_1 = 1$. However, by the minimality of m, we must then have $f_i = f_i^{(1)}$ for all i, and so each $f_i \in \mathbb{F}_q(t)$. Hence each $f_i = 0$ by linear independence over $\mathbb{F}_q(t)$.

5.1.3. *Fundamental matrices.* Given the inequality in (5.1.2.2), we can define a fundamental matrix for $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ to be a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ that satisfies,

$$\Psi^{(-1)} = \Phi \Psi.$$

Necessarily in this case the columns of Ψ form a basis of $\mathrm{Sol}(\Phi)$.

5.1.4. *Comparisons with Theorem 4.4.1.* Suppose that

$$\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$$

satisfies $\det \Phi = c(t-\theta)^s$, where $c \in \overline{k}^{\times}$ and $s \geq 0$. Suppose also that

$$\Psi \in \mathrm{GL}_r(\mathbb{L}) \cap \mathrm{Mat}_r(\mathbb{T})$$

is a fundamental matrix for Φ. It is not readily apparent from this data that Theorem 4.4.1 applies to the columns of Ψ because the entries of Ψ are not known to be in $\mathbb{E}$. However, the following lemma settles the matter. See Prop. 3.1.3 of [Anderson-Brownawell-Papanikolas].

Proposition 5.1.5. *Suppose that* $\Phi \in \mathrm{Mat}_{r\times r}(\overline{k}[t])$ *and* $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{T})$ *satisfy*

$$\det \Phi(0) \neq 0, \quad \psi^{(-1)} = \Phi \psi.$$

Then $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{E})$.

The rest of this section will focus on examples of fundamental matrices Ψ of particular interest. Moreover, as we wish to apply Theorem 4.4.1, the entries of the matrix

$$\Psi(\theta) \in \mathrm{Mat}_r(\mathbb{C}_\infty)$$

will be of particular interest to us. The theorem applies to the entries of $\Psi(\theta)$, but we will see that the entries of $\Psi(\theta)^{-1}$ are slightly cleaner.

5.2. **The Carlitz period and $\Omega(t)$.** In this example,

$$r = 1, \quad \Phi = t - \theta, \quad \Psi = \Omega(t).$$

We have already seen that

$$\Omega^{(-1)} = (t - \theta)\Omega.$$

We have also observed that $\Omega(\theta) = -\frac{1}{\pi_q}$, and so

$$\Omega(\theta)^{-1} = -\pi_q.$$

5.3. **Carlitz logarithms.**

5.3.1. *Definitions.* Recall from §2.4.1 that the Carlitz exponential function has the form

$$\exp_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q)\cdots(\theta^{q^i} - \theta^{q^{i-1}})}.$$

Its inverse is the Carlitz logarithm, which has a power series expansion

$$\log_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2})\cdots(\theta - \theta^{q^i})},$$

which converges for all $z \in \mathbb{C}_\infty$ with $|z|_\infty < |\theta|_\infty^{q/(q-1)}$. Like the Carlitz exponential function, the Carlitz logarithm has a functional equation,

$$\theta \log_C(z) = \log_C(\theta z) + \log_C(z^q), \tag{5.3.1.1}$$

which is valid as long as all three terms are defined. Also from the discussion in §4.2.2, we see

$$\log_C(\zeta_\theta) = \frac{\pi_q}{\theta}, \tag{5.3.1.2}$$

as a particular example of a Carlitz logarithm of an element in $\overline{k}$.

5.3.2. *Difference equations for Carlitz logarithms.* Now let $\alpha \in \overline{k}$ with $|\alpha|_\infty < |\theta|_\infty^{q/(q-1)}$. We are interested in properties (transcendental and otherwise) of the logarithm

$$\log_C(\alpha) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2})\cdots(\theta - \theta^{q^i})} \in \mathbb{C}_\infty.$$

Where this fits in with difference equations is through the function (see §6 of [Papanikolas 2008]),

$$L_\alpha(t) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t - \theta^q)(t - \theta^{q^2})\cdots(t - \theta^{q^i})},$$

which one can show is an element of $\mathbb{T}$. Of particular interest is that

$$L_\alpha(\theta) = \log_C(\alpha)$$

and

$$L_\alpha^{(-1)} = \alpha^{(-1)} + \frac{L_\alpha}{t - \theta}. \tag{5.3.2.1}$$

Thus if we let

$$\Phi = \begin{bmatrix} t-\theta & 0 \\ \alpha^{(-1)}(t-\theta) & 1 \end{bmatrix} \in \mathrm{Mat}_2(\overline{k}[t]),$$

then a fundamental matrix for Ψ is

$$\Psi = \begin{bmatrix} \Omega & 0 \\ \Omega L_\alpha & 1 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{E}).$$

We see that

$$\Psi(\theta) = \begin{bmatrix} -1/\pi_q & 0 \\ -\log_C(\alpha)/\pi_q & 1 \end{bmatrix}$$

and that

$$\Psi(\theta)^{-1} = \begin{bmatrix} -\pi_q & 0 \\ -\log_C(\alpha) & 1 \end{bmatrix}.$$

Note: Theorem 4.4.1 applies to the columns of $\Psi(\theta)$, but as we shall see $\Psi(\theta)^{-1}$ is more naturally the period matrix of the associated t-motive in this case. See §6.

5.4. **Carlitz polylogarithms and zeta values.** We take a short detour to discuss tensor powers of the Carlitz module and their associated exponential functions and periods. Ultimately Carlitz polylogarithms and zeta values come into the mix. The primary reference is [Anderson-Thakur 1990].

5.4.1. *Carlitz tensor powers.* The n-th tensor power of the Carlitz module the n-dimensional t-module $C^{\otimes n}$, which is defined by an $\mathbb{F}_q$-algebra homomorphism,

$$C^{\otimes n} : \mathbb{F}_q[t] \to \mathrm{Mat}_n(\mathbb{C}_\infty[F]),$$

where

$$C^{\otimes n}(t) = \theta \cdot \mathrm{Id} + N + A \cdot F$$

for

$$N = \begin{bmatrix} 0 & 1 & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}.$$

What is not yet apparent is why this should be considered a tensor power of C; however, that will be one of the subjects of §6.

5.4.2. *The exponential function for $C^{\otimes n}$.* The exponential function of $C^{\otimes n}$ is a vector valued function of the form

$$\exp_n \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \sum_{i=1}^{\infty} D_i \begin{bmatrix} z_1^{q^i} \\ \vdots \\ z_n^{q^i} \end{bmatrix}, \quad D_i \in \mathrm{Mat}_n(\mathbb{F}_q(\theta)).$$

The complete description of the coefficients D_i can be found in [Anderson-Thakur 1990] (see also §5.10 of [Goss] or §7.6 of [Thakur]). The inverse of $\exp_n$, the logarithm $\log_n$, is similarly defined. Of some importance to us are the specializations

$$\exp_n \begin{bmatrix} z \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \exp_C^{[n]}(z) \\ \vdots \\ \vdots \end{bmatrix},$$

where $\exp_C^{[n]}(z)$ is the single variable function,

$$\exp_C^{[n]}(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{\bigl[(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})\bigr]^n};$$

and

$$\log_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \\ \log_C^{[n]}(z) \end{bmatrix},$$

where $\log_C^{[n]}(z)$ is the single variable function,

$$\log_C^{[n]}(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{\bigl[(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})\bigr]^n}$$

The function $\log_C^{[n]}(z)$ is also called the *n-th Carlitz polylogarithm.*

In a similar way to our approach to the Carlitz logarithm, if we define for $\alpha \in \overline{k}$ with $|\alpha|_\infty < |\theta|_\infty^{qn/(q-1)}$,

$$L_{\alpha,n}(t) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{\bigl[(t - \theta^q)(t - \theta^{q^2}) \cdots (t - \theta^{q^i})\bigr]^n},$$

then

$$L_{\alpha,n}(\theta) = \log_C^{[n]}(\alpha).$$

Furthermore, $L_{\alpha,n}$ satisfies the functional equation

$$L_{\alpha,n}^{(-1)} = \alpha^{(-1)} + \frac{L_{\alpha,n}}{(t-\theta)^n}.$$

See [Chang-Yu 2007] for more details on the function $L_{\alpha,n}$. Now if we let

$$\Phi = \begin{bmatrix} (t-\theta)^n & 0 \\ \alpha^{(-1)}(t-\theta)^n & 1 \end{bmatrix} \in \mathrm{Mat}_2(\overline{k}[t]),$$

then a fundamental matrix for Ψ is

$$\Psi = \begin{bmatrix} \Omega^n & 0 \\ \Omega^n L_{\alpha,n} & 1 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{E}).$$

We see that

$$\Psi(\theta) = \begin{bmatrix} (-1)^n/\pi_q^n & 0 \\ (-1)^n \log_C^{[n]}(\alpha)/\pi_q^n & 1 \end{bmatrix}$$

and that

$$\Psi(\theta)^{-1} = \begin{bmatrix} (-\pi_q)^n & 0 \\ -\log_C^{[n]}(\alpha) & 1 \end{bmatrix}.$$

5.4.3. *Carlitz zeta values.* For an integer $s \geq 1$, we define the Carlitz zeta value

$$\zeta_C(s) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ a \text{ monic}}} \frac{1}{a^s} \in k_\infty.$$

These zeta values share many properties with special values of the Riemann zeta function. For example, if $(q-1) \mid s$, then there is $r_s \in \mathbb{F}_q(\theta)$ so that

$$\zeta_C(s) = r_s \pi_q^s.$$

One considers these to be "even" Carlitz zeta values, and it has been proved by Anderson, Thakur, and Yu that "odd" zeta values (i.e., the remaining ones) are not $\overline{k}$-multiples of π_q^s (see [Anderson-Thakur 1990] and [Yu 1991]). A remarkable result of [Anderson-Thakur 1990] is that it is possible to find $h_0, \dots, h_\ell \in \mathbb{F}_q[\theta]$ so that

$$\zeta_C(n) = \frac{1}{\Gamma_n} \sum_{i=0}^{\ell} h_i L_{\theta^i,n}(\theta), \tag{5.4.3.1}$$

where $\Gamma_n \in \mathbb{F}_q[\theta]$ is a Carlitz factorial (see [Goss] or [Thakur]).

There are several consequences of (5.4.3.1). First Anderson and Thakur used this formula to show that there are points in $C^{\otimes n}(\overline{k})$ whose logarithms (via $\log_n$) can be explicitly related to $\zeta_C(n)$. Second is that if we let, as in [Chang-Yu 2007],

$$\Phi = \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ (\theta^0)^{(-1)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\theta^\ell)^{(-1)}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{Mat}_{\ell+2}(\overline{k}[t]),$$

then Φ has a fundamental matrix

$$\Psi = \begin{bmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n L_{\theta^0,n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n L_{\theta^\ell,n} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{Mat}_{\ell+2}(\mathbb{E}).$$

Chang and Yu use this description to identify all algebraic relations among Carlitz zeta values, as we will see later in the notes.

5.5. **Drinfeld modules of rank $r \geq 2$.** Suppose that $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ is a Drinfeld module of rank 2 for $\mathbb{F}_q[t]$. By a change of variables, we can assume

$$\rho(t) = \theta + \kappa F + F^2, \quad \kappa \in \overline{k}.$$

As discussed in §2.3, there is a unique exponential power series,

$$\exp_\rho(z) = \sum_{i=0}^{\infty} \alpha_i z^{q^i}, \quad \alpha_i \in \overline{k},\ \alpha_0 = 1,$$

satisfying the functional equation

$$\exp_\rho(\theta z) = \rho(t)(\exp_\rho(z)) = \theta \exp_\rho(z) + \kappa \exp_\rho(z)^q + \exp_\rho(z)^{q^2}.$$

What follows can also be made explicit for Drinfeld modules of arbitrary rank r, but we will stick with $r = 2$ for this example.

5.5.1. *An associated system of difference equations.* (See also [Pellarin 2007].) For fixed $u \in \overline{k_\infty}$, we define a power series

$$s_u(t) := -\sum_{i=0}^{\infty} \exp_\rho\left(\frac{u}{\theta^{i+1}}\right) t^i = \sum_{i=0}^{\infty} \frac{\alpha_i u^{q^i}}{t - \theta^{q^i}} \in \mathbb{T}.$$

We note that $s_u(t)$ is a meromorphic function on $\mathbb{C}_\infty$ that has simples poles at $t = \theta, \theta^q, \ldots$, with residues $u, \alpha_1 u^q, \ldots$. Since $\rho(t)(\exp_\rho(u/\theta^{i+1})) = \exp_\rho(u/\theta^i)$, we have

$$\kappa s_u^{(1)}(t) + s_u^{(2)}(t) = (t - \theta) s_u(t) - \exp_\rho(u). \tag{5.5.1.1}$$

Since $s_u^{(j)}(t)$ converges away from $\{\theta^{q^j}, \theta^{q^{j+1}}, \ldots\}$ and the residue of $s_u(t)$ at $t = \theta$ is u, it follows that when we evaluate at $t = \theta$, we have

$$\kappa s_u^{(1)}(\theta) + s_u^{(2)}(\theta) = u - \exp_\rho(u). \tag{5.5.1.2}$$

Fix generators $\omega_1, \omega_2 \in \overline{k_\infty}$ of the period lattice of ρ; that is, the kernel of $\exp_\rho(z)$ is $\mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2$. Now let

$$s_1(t) = s_{\omega_1}(t), \quad s_2(t) = s_{\omega_2}(t),$$

and set

$$\widetilde{\Phi} := \begin{bmatrix} 0 & t - \theta \\ 1 & -\kappa \end{bmatrix}, \quad \widetilde{\Psi} := \begin{bmatrix} s_1 & s_1^{(1)} \\ s_2 & s_2^{(1)} \end{bmatrix}.$$

Then the functional equation (5.5.1.1) implies that

$$\widetilde{\Psi}^{(1)} = \widetilde{\Psi}\widetilde{\Phi},$$

which is not quite the type of functional equation we need. However, if we let

$$\Phi = \begin{bmatrix} 0 & 1 \\ t - \theta & -\kappa^{(-1)} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix} \left(\widetilde{\Psi}^{(1)}\right)^{-1},$$

then

$$\Psi^{(-1)} = \Phi\Psi.$$

It is possible to show that $\widetilde{\Psi} \in \mathrm{GL}_2(\mathbb{T})$ and so the entries of Ψ are in $\mathbb{T}$ (and in fact in $\mathbb{E}$ by Prop. 5.1.5).

5.5.2. *Specializing at $t = \theta$.* As in previous examples, we evaluate $\Psi(\theta)^{-1}$:

$$\begin{aligned}\Psi(\theta)^{-1} &= \widetilde{\Psi}^{(1)}(\theta)\begin{bmatrix}\kappa & 1\\ 1 & 0\end{bmatrix}\\ &= \begin{bmatrix}\kappa s_1^{(1)}(\theta) + s_1^{(2)}(\theta) & s_1^{(1)}(\theta)\\ \kappa s_2^{(1)}(\theta) + s_2^{(2)}(\theta) & s_2^{(1)}(\theta)\end{bmatrix}\\ &= \begin{bmatrix}\omega_1 & s_1^{(1)}(\theta)\\ \omega_2 & s_2^{(1)}(\theta)\end{bmatrix},\end{aligned}$$

where the last equality follows from (5.5.1.2). It turns out that

$$s_j^{(1)}(\theta) = -\sum_{i=1}^{\infty} \exp_\rho\left(\frac{\omega_j}{\theta^{i+1}}\right)^q \theta^i = -F_1(\omega_j), \quad j = 1,\ 2,$$

where F_1 is the quasi-periodic function associated to $\delta(t) = \tau$ for ρ, as defined in §3.5.2. Thus, the upshot is that

$$\Psi(\theta)^{-1} = \begin{bmatrix}\omega_1 & \eta_1\\ \omega_2 & \eta_2\end{bmatrix},$$

where ω_1, ω_2 generate the period lattice for ρ and η_1, η_2 are their corresponding quasi-periods.

5.6. **Gamma values.** Theorem 4.4.1 was originally formulated in [Anderson-Brownawell-Papanikolas] to address the problem of algebraic relations among special Γ-values. Unfortunately, the story is too long to give it proper justice here, so we will be brief.

The geometric Γ-function over $\mathbb{F}_q[\theta]$ is defined by the infinite product

$$\Gamma(z) = \frac{1}{z} \prod_{\substack{n \in \mathbb{F}_q[\theta]\\ n \text{ monic}}} \left(1 + \frac{z}{n}\right)^{-1}, \quad z \in \mathbb{C}_\infty.$$

This function was studied extensively by Thakur [Thakur 1991]. The special Γ-values are defined to be

$$\{\Gamma(z) \mid z \in \mathbb{F}_q(\theta) \setminus \mathbb{F}_q[\theta]\}.$$

What is interesting about these values is that, much in the same way as the classical Γ-function, the functional equations for this Γ-function induce many $\bar{k}$-algebraic relations among them. The main theorem of [Anderson-Brownawell-Papanikolas] was that these relations account for all algebraic relations.

The connection with Drinfeld modules was first observed by Thakur in several examples. For example,

$$\Gamma\left(\frac{1}{\theta}\right) = \frac{\theta}{\sqrt[q-1]{\theta\zeta_\theta}}\, \pi_{\zeta_\theta},$$

where π_{ζ_θ} is the Carlitz period for the polynomial ring $\mathbb{F}_q[\zeta_\theta]$ (see §6.1 of [Brownawell-Papanikolas 2002]). Then Sinha proved in [Sinha 1997a] that every value

$$\Gamma\left(\frac{a}{f}\right), \quad a, f \text{ both } \textit{monic},$$

arises as a period of a specially constructed t-module by defining and using Coleman functions (based on work in [Coleman 1988] and extended in [Anderson 1992]). In [Brownawell-Papanikolas 2002] it is shown that the remaining special Γ-values arise as quasi-periods. Building on the work of Sinha, it is shown in [Anderson-Brownawell-Papanikolas] how everything fits in with specially defined difference equations, at which point Theorem 4.4.1 can be applied.

6. t-Motives

6.1. **Anderson t-motives.** In this section we will define *Anderson t-motives* and show how they are related to Drinfeld modules (and t-modules) and how they are connected to the difference equations we saw in the last section. A word to the wise is that these t-motives are slightly different from the ones defined in [Anderson 1986], although they have the same flavor. The references for these types of t-motives are §4 of [Anderson-Brownawell-Papanikolas] (where they are called "dual t-motives") and §3.4 of [Papanikolas 2008].

6.1.1. *The ring* $\overline{k}[t,\boldsymbol{\sigma}]$. The ring $\overline{k}[t,\boldsymbol{\sigma}]$ is the polynomial ring t and $\boldsymbol{\sigma}$ with coefficients in $\overline{k}$ subject to the following relations,

$$tc = ct, \quad t\boldsymbol{\sigma} = \boldsymbol{\sigma}t, \quad \boldsymbol{\sigma}c = c^{1/q}\boldsymbol{\sigma}, \quad \forall\ c \in \overline{k}.$$

In this way for any $f \in \overline{k}[t]$,

$$\boldsymbol{\sigma}f = f^{(-1)}\boldsymbol{\sigma} = \sigma(f)\boldsymbol{\sigma}.$$

6.1.2. *Definition.* An *Anderson t-motive* M is a left $\overline{k}[t,\boldsymbol{\sigma}]$-module such that

- M is free and finitely generated over $\overline{k}[t]$;
- M is free and finitely generated over $\overline{k}[\boldsymbol{\sigma}]$;
- $(t-\theta)^n\mathsf{M} \subseteq \boldsymbol{\sigma}\mathsf{M}$ for all $n \gg 0$.

Given an Anderson t-motive M, if the entries of $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(\mathsf{M})$ is a $\overline{k}[t]$-basis for M, then there is a matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ so that

$$\boldsymbol{\sigma}\mathbf{m} = \Phi\mathbf{m}.$$

Since a power of $t-\theta$ annihilates $\mathsf{M}/\boldsymbol{\sigma}\mathsf{M}$, we have

$$\det\Phi = c(t-\theta)^s,$$

for some $c \in \overline{k}^{\times}$, where s is the rank of M as a $\overline{k}[\boldsymbol{\sigma}]$-module.

6.1.3. *Category of Anderson t-motives.* This category is defined as follows:

- Objects are Anderson t-motives;
- Morphisms are morphisms of left $\overline{k}[t, \boldsymbol{\sigma}]$-modules.

What is quite useful is the following theorem.

Theorem 6.1.4 ([Anderson 1986])**.** *The category of Anderson t-motives contains the category of abelian t-modules as a full subcategory.*

It is not so important at the moment what an "abelian" t-module is (the definition is in [Anderson 1986] and in [Goss]), but it suffices to say that all Drinfeld modules are abelian, as are all tensor powers of the Carlitz module and many other natural t-modules. In fact the essential image of abelian t-modules in the category of Anderson t-motives can be described explicitly.

6.1.5. *The functor from Drinfeld modules to Anderson t-motives.* In this form, the definition is due to Anderson. Suppose we have a Drinfeld module $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ given by

$$\rho(t) = \theta + a_1 F + \cdots + a_r F^r.$$

We define the Anderson t-motive associated to ρ in the following way. Let $\mathsf{M} = \overline{k}[\boldsymbol{\sigma}]$. Since M has a natural left $\overline{k}[\boldsymbol{\sigma}]$-module structure, it suffices to define the $\overline{k}[t]$-module structure. For this, we set for $h \in \overline{k}[\boldsymbol{\sigma}]$,

$$t \cdot h(\boldsymbol{\sigma}) := \rho(t) \diamond h(\boldsymbol{\sigma}),$$

where the $\diamond$-product is defined by

$$f(F) \diamond g(\boldsymbol{\sigma}) := \sum_i \sum_j a_i^{(-i-j)} b_j \boldsymbol{\sigma}^{i+j},$$

for $f = \sum_i a_i F^i$, $g = \sum_j b_j \boldsymbol{\sigma}^j$. That this definition extends to a well-defined $\overline{k}[t]$-module operation which makes M into an Anderson t-motive is somewhat straightforward and left to the reader.

In a pleasant way we can also recover the Drinfeld module ρ from M. Indeed, we find that

$$\frac{\mathsf{M}}{(\boldsymbol{\sigma} - 1)\mathsf{M}} \cong \rho,$$

as $\mathbb{F}_q[t]$-modules in a natural way. We will observe this identification in the next two examples.

6.2. t-Motive examples.

6.2.1. *The Carlitz motive.* By using the description above we can define the Anderson t-motive C that corresponds to the Carlitz module as follows. We let $\mathsf{C} = \overline{k}[t]$ and define a left $\overline{k}[\boldsymbol{\sigma}]$-module structure on C by setting

$$\boldsymbol{\sigma}(f) = (t - \theta) f^{(-1)}, \quad \forall f \in \mathsf{C}.$$

To see that this comes from the Carlitz module, we first note that

$$\mathsf{C}/(\boldsymbol{\sigma} - 1)\mathsf{C} \cong \overline{k},$$

as $\mathbb{F}_q$-vector spaces. We now check that the t-action on the left corresponds to the $C(t)$-action on the right. For $x \in \overline{k}$,

$$\begin{aligned} tx = \theta x + (t-\theta)x &= \theta x + \boldsymbol{\sigma}(x^q) \\ &= \theta x + x^q + (\boldsymbol{\sigma} - 1)(x^q) \\ &= C(t)(x) + (\boldsymbol{\sigma} - 1)(x^q). \end{aligned}$$

6.2.2. *Anderson t-motives for rank 2 Drinfeld modules.* Let ρ be a rank 2 Drinfeld module given by

$$\rho(t) = \theta + \kappa F + F^2, \quad \kappa \in \overline{k}.$$

Again after using the functor from Drinfeld modules to t-motives, we arrive at the following construction. We let $\mathsf{M} = \mathrm{Mat}_{1\times 2}(\overline{k}[t])$, and define the action of $\overline{k}[\boldsymbol{\sigma}]$ on M by

$$\boldsymbol{\sigma}(f,g) := \left[f^{(-1)}, g^{(-1)}\right] \begin{bmatrix} 0 & 1 \\ t-\theta & -\kappa^{(-1)} \end{bmatrix}.$$

We check that for $a \in \overline{k}$,

$$\begin{aligned} (\boldsymbol{\sigma} - 1)[a,0] &= [-a, a^{(-1)}], \\ (\boldsymbol{\sigma}^2 - 1)[a,0] &= [a^{(-2)}(t-\theta) - a, -\kappa^{(-1)}a^{(-2)}]. \end{aligned}$$

For $x \in \overline{k}$, we see that

$$\begin{aligned} t[x,0] = [tx,0] &= [tx + \kappa x^q, -\kappa^{(-1)}x] + [-\kappa x^q, \kappa^{(-1)}x] \\ &= [tx + \kappa x^q, -\kappa^{(-1)}x] + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] \\ &= [\theta x + \kappa x^q + x^{q^2}, 0] + [(t-\theta)x - x^{q^2}, -\kappa^{(-1)}x] + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] \\ &= [\theta x + \kappa x^q + x^{q^2}, 0] + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] + (\boldsymbol{\sigma}^2 - 1)[x^{q^2}, 0]. \end{aligned}$$

Thus we see that the action of t on $\mathsf{M}/(\boldsymbol{\sigma} - 1)\mathsf{M}$ is the same as the action of $\rho(t)$ on $\overline{k}$.

6.3. **Rigid Analytic Triviality.** In the examples above, we observed the following chain of constructions:

$$\left\{\begin{matrix}\text{Drinfeld module} \\ \text{or } t\text{-module } \rho\end{matrix}\right\} \Longrightarrow \left\{t\text{-motive } \mathsf{M}\right\} \Longrightarrow \left\{\begin{matrix}\Phi \in \mathrm{Mat}_r(\overline{k}[t]) \\ \text{representing } \boldsymbol{\sigma}\end{matrix}\right\}$$

$$\stackrel{(*)}{\Longrightarrow} \left\{\begin{matrix}\Psi \in \mathrm{Mat}_r(\mathbb{E}), \\ \Psi^{(-1)} = \Phi\Psi\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix}\Psi(\theta)^{-1} \text{ provides} \\ \text{periods of } \rho\end{matrix}\right\}.$$

One thing that is not clear is whether or not this construction works for all Drinfeld modules and t-modules. Indeed it does not in all cases, and the sticking point is the construction $(*)$. If $(*)$ does hold, then we say that our t-motive M is *rigid analytically trivial.* That is, if the matrix Φ that represents multiplication by $\boldsymbol{\sigma}$ on a given t-motive M has a fundamental matrix with entries in $\mathbb{E}$ (or even simply $\mathbb{T}$), then M is rigid analytically trivial.

The name comes from the fact that then the space $\mathsf{M} \otimes_{\overline{k}[t]} \mathbb{T}$ naturally decomposes into a direct sum of copies of $\mathbb{T}$ on which $\boldsymbol{\sigma}$ acts simply via σ. In fact the entries of

$$\Psi^{-1}\mathbf{m}$$

are all fixed by $\boldsymbol{\sigma}$.

A deep theorem of Anderson (see [Anderson 1986]) proves the following equivalence,

$$\left\{\begin{matrix}\text{Drinfeld module or } t\text{-} \\ \text{module is uniformizable}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}t\text{-motive } \mathsf{M} \text{ is rigid} \\ \text{analytically trivial}\end{matrix}\right\}.$$

Here uniformizable refers to whether or not the exponential function is surjective, as mentioned briefly in §3.1.2. However, as we saw in §2.3, the exponential functions of Drinfeld modules are always surjective, and so their associated Anderson t-motives are always rigid analytically trivial. Rigid analytic triviality is also closed under direct sums and taking submodules.

7. Difference Galois groups and algebraic independence

In this section we summarize the main results of [Papanikolas 2008], which shows that the transcendence degree of a period matrix $\Psi(\theta)^{-1}$ over $\overline{k}$ is the same as the dimension of an associated Galois group.

7.1. **Galois group construction.**

7.1.1. *Preliminaries.* To begin, we can set things up in a somewhat general setting. As in §5.1, we start with fields $K \subseteq L$ together with an automorphism $\sigma : L \xrightarrow{\sim} L$ such that,

- $\sigma(K) = K$;
- L/K is separable;
- $L^\sigma = K^\sigma =: E$.

We also suppose that we have a matrices $\Phi \in \mathrm{GL}_r(K)$ and $\Psi \in \mathrm{GL}_r(L)$ so that Ψ is a fundamental matrix for Φ. That is,

$$\sigma(\Psi) = \Phi\Psi.$$

Let $X = (X_{ij})$ denote an $r \times r$ matrix of variables. We define a K-algebra homomorphism,

$$\nu = (X_{ij} \mapsto \Psi_{ij}) : K[X, 1/\det X] \to L,$$

and we let $\Sigma = \operatorname{im} \nu$. We let Λ be the fraction field of Σ. In this way,

$$\Sigma = K[\Psi, 1/\det \Psi], \quad \Lambda = K(\Psi).$$

Because of the functional equation for Φ and Ψ, it follows that both Σ and Λ are invariant under σ. We further assume that

- K is algebraically closed in Λ.

The particular example we have in mind of course is $(E, K, L) = (\mathbb{F}_q(t), \overline{k}(t), \mathbb{L})$.

7.1.2. *The Galois group* Γ. For a ring R, let $\mathrm{GL}_{r/R}$ denote the R-group scheme for GL_r, whose coordinate ring is $R[X, 1/X]$. For an R-scheme Y and an R-algebra S, we let $Y_S = Y \times_R S$ denote the extension of Y to and S-scheme.

Let Z be the smallest closed K-subscheme of $\mathrm{GL}_{r/K}$ such that $\Psi \in Z(L)$. Certainly,

$$Z \cong \operatorname{Spec} \Sigma.$$

Now set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$ to be the matrices such that

$$(\Psi_1)_{ij} = \Psi_{ij} \otimes 1, \quad (\Psi_2)_{ij} = 1 \otimes \Psi_{ij},$$

and set $\widetilde{\Psi} = \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$. We define an E-algebra map,

$$\mu = (X_{ij} \mapsto \widetilde{\Psi}_{ij}) : E[X, 1/\det X] \to L \otimes_K L, \tag{7.1.2.1}$$

which in this way defines a closed E-subscheme Γ of $\mathrm{GL}_{r/E}$ that is characterized as the smallest such subscheme with $\widetilde{\Psi} \in \Gamma(L \otimes_K L)$.

To show that Γ is in fact a group scheme over E, we start with the following working hypotheses, which arrive through analogies with the Galois theory of differential equations.

- $Z(\overline{L})$ is a right coset for Γ.
- Since $\Psi \in Z(\overline{L})$, it follows that we should have $\Gamma(\overline{L}) = \Psi^{-1}Z(\overline{L})$.
- The isomorphism above should induce an isomorphism

$$(u, v) \mapsto (u, u^{-1}v) : Z \times Z \xrightarrow{\sim} Z \times \Gamma.$$

- Everything should be done in such a way as to be defined over the smallest field possible (say E, K, or L).

To realize that these working hypotheses are indeed true, one proves the following two propositions. For the proofs of both, see §4.2 of [Papanikolas 2008].

Proposition 7.1.3. *Define a morphism of affine L-schemes $Z_L \to \mathrm{GL}_{r/L}$ so that on points $u \mapsto \Psi^{-1}u$ for $u \in Z(\overline{L})$. Then this map factors through an isomorphism $Z_L \to \Gamma_L$ of affine L-schemes.*

Proposition 7.1.4. *Define a morphism of affine K-schemes $Z \times Z \to Z \times \mathrm{GL}_{r/K}$ so that on points $(u, v) \mapsto (u, u^{-1}v)$ for u, $v \in Z(\overline{K})$. Then this map factors through an isomorphism $Z \times Z \to Z \times \Gamma_K$ of affine K-schemes.*

From these two propositions it is possible to prove via the Yoneda lemma that Γ is in fact a group scheme over E and that Z is a Γ_K-torsor. We summarize these results and some other consequences in the following theorem.

Theorem 7.1.5 ([Papanikolas 2008])**.** *Maintaining the definitions above, the following statements hold.*

(a) *Γ is a closed E-subgroup scheme of $\mathrm{GL}_{r/E}$.*
(b) *Z is stable under right-multiplication by Γ_K and is a Γ_K-torsor.*
(c) *The K-scheme Z is absolutely irreducible and is smooth over $\overline{K}$.*
(d) *The E-scheme Γ is absolutely irreducible and is smooth over $\overline{E}$.*
(e) *The dimension of Γ over F is equal to the transcendence degree of Λ over K.*

(f) *If every element of* $\overline{E}$ *is fixed by some power of* σ*, then the elements of* Λ *fixed by* $\Gamma(\overline{E})$ *are precisely* K*.*

7.1.6. *Remarks.* In the theorem above, (a) and (b) follow somewhat generally as long as (E, K, L) satisfy the first three conditions in §7.1.1. The other parts (c)–(f) rely heavily on the condition that K be algebraically closed in Λ. See [Papanikolas 2008] for more details.

7.2. **Connections with t-motives.**

7.2.1. *The Tannakian category of t-motives.* It will take us somewhat far afield to go through all of the details on this topic. However, it is important to mention what happens. Given two rigid analytically trivial Anderson t-motives, M_1 and M_2, we can form their direct sum

$$\mathsf{M}_1 \oplus \mathsf{M}_2,$$

which is naturally also a rigid analytically trivial Anderson t-motive. We can also form their tensor product,

$$\mathsf{M}_1 \otimes_{\overline{k}[t]} \mathsf{M}_2,$$

and if we specify that $\boldsymbol{\sigma}$ acts diagonally on this space (i.e., $\boldsymbol{\sigma}(m_1 \otimes m_2) := \boldsymbol{\sigma}(m_1) \otimes \boldsymbol{\sigma}(m_2)$), then one can show that $\mathsf{M}_1 \otimes_{\overline{k}[t]} \mathsf{M}_2$ is also a rigid analytically trivial Anderson t-motive (see [Anderson 1986]). We should note that the tensor powers of the Carlitz module discussed in §5.4.1 are defined in exactly this way.

If we take a t-motive M and tensor with $\overline{k}(t)$, we obtain a finite dimensional $\overline{k}(t)$-vector space

$$M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathsf{M}.$$

Then M carries the natural structure of a left $\overline{k}(t)[\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}]$-module with the following properties:

- M is finite dimensional as a $\overline{k}(t)$-vector space;
- multiplication by σ on M is represented by a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ that has a fundamental matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$.

(By allowing denominators, we lose control over the determinant of Φ.) In §3 of [Papanikolas 2008], we see that the category $\mathcal{R}$ of modules satisfying these properties form a neutral Tannakian category over $\mathbb{F}_q(t)$. That is, all of the operations on group representations over a field (direct sums, tensor products, duals, kernels, cokernels, internal Hom's, etc.) can be performed $\mathcal{R}$. The fiber functor is defined by

$$\omega(M) = (\mathbb{L} \otimes_{\overline{k}(t)} M)^{\sigma},$$

and it can also be realized as the $\mathbb{F}_q(t)$-span of the entries of $\Psi^{-1}\mathbf{m}$, if $\mathbf{m}$ is the basis of our t-motive. See [Breen] or [Deligne-Milne-Ogus-Shih 1982] for more information on Tannakian categories.

We call the category $\mathcal{T}$ of *t-motives* the smallest (strictly full) Tannakian subcategory of $\mathcal{R}$ that contains the Anderson t-motives. For a t-motive M, we can form the category $\mathcal{T}_M$, which is the smallest Tannakian subcategory of $\mathcal{T}$ containing M.

Now Tannakian duality asserts that any neutral Tannakian category over a field E is naturally isomorphic to the category of representations of some affine group scheme over E.

Thus to each t-motive M we can associate a group scheme Γ_M over $\mathbb{F}_q(t)$. In fact, from general principles it turns out that $\Gamma_M \subseteq \mathrm{GL}_{r/\mathbb{F}_q(t)}$. We then have the following theorem, see Thm. 4.5.10 of [Papanikolas 2008].

Theorem 7.2.2. *Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by $\boldsymbol{\sigma}$ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. Then the Galois group Γ_Ψ associated to the difference equations*

$$\Psi^{(-1)} = \Phi\Psi$$

is naturally isomorphic to the group Γ_M associated to M via Tannakian duality.

7.3. **Galois groups and transcendence.** The following theorem is the main theorem (Thm. 5.2.2) of [Papanikolas 2008]. It determines exactly the transcendence degree of the period matrix of a t-motive. We will give a sketch of the proof, which relies heavily on Theorem 4.4.1. Then in §8, we will revisit our various examples for applications to algebraic independence.

Theorem 7.3.1. *Let M be a t-motive, and let Γ_M be its associated group via Tannakian duality. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by $\boldsymbol{\sigma}$ on M and that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^{\times}$. Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$. Finally let*

$$L = \overline{k}(\Psi(\theta)) \subseteq \overline{k_\infty}.$$

Then

$$\mathrm{tr.\ deg}_{\overline{k}} L = \dim \Gamma_M.$$

7.3.2. *Remark.* If M arises from an Anderson t-motive M that is rigid analytically trivial, then the hypotheses of the theorem are automatically satisfied.

7.3.3. *Sketch of the proof.* The idea is the following. Fix $d \geq 1$. For each $n \geq 1$, the entries of the Kronecker product matrix $\Psi^{\otimes n}$ are all degree n monomials in the entries of Ψ. We let $\overline{\psi}$ be the column vector whose entries are the concatenation of 1 and each of the columns of $\Psi^{\otimes n}$ for $n \leq d$. Let $\overline{\Phi}$ be the block diagonal matrix,

$$\overline{\Phi} = [1] \oplus \Phi^{\oplus r} \oplus (\Phi^{\otimes 2})^{\oplus r^2} \oplus \cdots \oplus (\Phi^{\otimes d})^{\oplus r^d}.$$

Then it is evident that

$$\overline{\psi}^{(-1)} = \overline{\Phi}\,\overline{\psi},$$

and by our hypotheses, Theorem 4.4.1 can apply. Moreover, any polynomial relations among the entries of $\Psi(\theta)$ will eventually appear as a linear relation among the entries of $\overline{\psi}(\theta)$, once d is large enough. The idea then is use Theorem 4.4.1 to show an equality,

$$\dim_{\overline{k}} Q_d = \dim_{\overline{k}(t)} S_d,$$

where

- Q_d is the $\overline{k}$-span of the entries of $\overline{\psi}(\theta)$;
- S_d is the $\overline{k}(t)$-span of the entries of ψ.

By doing this for each d, it follows that the transcendence degree of $\overline{k}(\Psi(\theta))$ over $\overline{k}$ is the same as the transcendence degree of $\overline{k}(t)(\Psi)$ over $\overline{k}(t)$.

8. Applications

In this section we will revisit the examples we have seen already and discuss recent results and progress toward algebraic independence.

8.1. **The Carlitz period.** We consider again the case $\Phi = t - \theta$ and $\Psi = \Omega$. We see from 6.2.1 that Φ represents multiplication by $\boldsymbol{\sigma}$ on the Carlitz motive C. Since Ω is a transcendental function over $\overline{k}(t)$, it follows that the Galois group $\Gamma_C \cong \Gamma_\Psi$ has dimension

$$\dim \Gamma_C = \mathrm{tr.\ deg}_{\overline{k}(t)}\ \overline{k}(t)(\Omega) = 1.$$

Thus $\Gamma_C \cong \mathrm{GL}_1 / \mathbb{F}_q(t) = \mathbb{G}_m / \mathbb{F}_q(t)$. For $\gamma \in \mathbb{G}_m(\mathbb{F}_q(t)) = \mathbb{F}_q(t)^\times$, the action of γ on $\overline{k}(t)(\Omega)$ is naturally defined by

$$\gamma : h(t, \Omega) \mapsto h(t, \Omega\gamma).$$

We should note that another way of interpreting this calculation is that C is a Tate object in the category of t-motives, whereby tensoring with C provides an equivalence of categories.

8.2. **Carlitz logarithms.** The main source for these results is [Papanikolas 2008]. However, Denis has also determined special cases of these results in [Denis 2006] using Mahler's method.

Let $\alpha_1, \dots, \alpha_r \in \overline{k}$, with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$. Our goal is to determine all algebraic relations over $\overline{k}$ among

$$\log_C(\alpha_1), \dots, \log_C(\alpha_r) \in \mathbb{C}_\infty.$$

We already know some possible algebraic relations, those which are generated from ones of the form in (5.3.1.1). The main result here will be that these are the only ones.

8.2.1. *Logarithms and t-motives.* As we saw in §5.3, we consider the matrix

$$\Phi = \begin{bmatrix} t-\theta & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t-\theta) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{(-1)}(t-\theta) & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{r+1}(\overline{k}(t)) \cap \mathrm{Mat}_{r+1}(\overline{k}[t]).$$

Now Φ represents multiplication by $\boldsymbol{\sigma}$ on an Anderson t-motive X, and X sits an exact sequence

$$0 \to C \to X \to \mathbf{1}^r \to 0,$$

where $\mathbf{1}$ is the trivial t-motive. (The underlying space for $\mathbf{1}$ is $\overline{k}(t)$, and the $\boldsymbol{\sigma}$-action on $\mathbf{1}$ coincides with the usual σ-action. The t-motive $\mathbf{1}$ does not come from an Anderson t-motive directly. It is "trivial" because it is the identity with respect to tensor products: $\mathbf{1} \otimes M \cong M$ for all t-motives M.) The t-motive X is also rigid analytically trivial, and as we have seen, the fundamental matrix for Φ is

$$\Psi = \begin{bmatrix} \Omega & 0 & \cdots & 0 \\ \Omega L_{\alpha_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega L_{\alpha_r} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T}) \cap \mathrm{Mat}_{r+1}(\mathbb{E}).$$

8.2.2. *The Galois group.* We are in the situation that Theorem 7.3.1 applies, but we need to calculate the Galois group of Ψ. To do this, we note that

$$\Gamma_\Psi \subseteq \left\{ \begin{bmatrix} * & 0 \\ * & I_r \end{bmatrix} \right\},$$

by (7.1.2.1). Thus, the coordinates of X are narrowed down to those in the first column, and we relabel them $X_0, \dots, X_r$. Now there is a natural map

$$\Gamma_\Psi \to \mathbb{G}_m,$$

given by projection onto the first coordinate. One shows that this map coincides with the projection

$$\Gamma_\Psi \twoheadrightarrow \Gamma_C,$$

and so is also surjective. Thus we have an exact sequence of group schemes,

$$0 \to V \to \Gamma_\Psi \twoheadrightarrow \mathbb{G}_m \to 1.$$

If we consider $\alpha \in \mathbb{G}_m(\overline{\mathbb{F}_q(t)})$ and lift it back to an element $\gamma \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$, then for any $\mu = \left[\begin{smallmatrix} 1 & 0 \\ v & I_r \end{smallmatrix}\right] \in V(\overline{\mathbb{F}_q(t)})$, we find that

$$\gamma^{-1}\mu\gamma = \begin{bmatrix} 1 & 0 \\ \alpha v & I_r \end{bmatrix} \in V(\overline{\mathbb{F}_q(t)}).$$

Thus $V(\overline{\mathbb{F}_q(t)})$ is a vector subspace of $\overline{\mathbb{F}_q(t)}^r$. Since V is smooth (because Γ_Ψ projects onto $\mathbb{G}_m$), it follows that the defining equations for V are linear forms in $X_1, \dots, X_r$, which can be defined over $\mathbb{F}_q(t)$. Hilbert's Theorem 90 then provides an exact sequence,

$$0 \to V(\mathbb{F}_q(t)) \to \Gamma_\Psi(\mathbb{F}_q(t)) \to \mathbb{G}_m(\mathbb{F}_q(t)) \to 1.$$

Using this description it is possible to write down explicit equations for Γ_Ψ (see Thm. 6.3.2 of [Papanikolas 2008]). We will describe this in an example momentarily. For now fix $b_0 \in \mathbb{F}_q(t)^\times \setminus \mathbb{F}_q^\times$ and choose a matrix

$$\gamma = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_r & 0 & \cdots & 1 \end{bmatrix} \in \Gamma_\Psi(\mathbb{F}_q(t)).$$

The defining equations for Γ_Ψ are given as follows.

Theorem 8.2.3. *With notation as above, the following statements hold.*

(a) *Suppose $F = c_1X_1 + \cdots + c_rX_r$, $c_1, \dots, c_r \in \mathbb{F}_q(t)$, is a defining linear form for V. Then*

$$G = (b_0 - 1)F - F(b_1, \dots, b_r)(X_0 - 1)$$

is a defining polynomial for Γ_Ψ. In particular,

$$(b_0(\theta) - 1)\sum_{i=1}^r c_i(\theta)\log_C(\alpha_i) - \sum_{i=1}^r c_i(\theta)b_i(\theta)\pi_q = 0.$$

(b) *Every k-linear relation among π_q, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$ is a k-linear combination of the relations from part (a).*

(c) *Let N be the k-linear span of π_q, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$. Then*
$$\dim \Gamma_\Psi = \dim_k N.$$

By Theorem 7.3.1, it follows that the transcendence degree of π_q, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$ over $\overline{k}$ is then equal to $\dim_k N$. This leads to the following result of [Papanikolas 2008].

Theorem 8.2.4 ([Papanikolas 2008], Thm. 6.4.2)**.** *Let $\lambda_1, \dots, \lambda_r \in \mathbb{C}_\infty$ satisfy $\exp_C(\lambda_i) \in \overline{k}$ for $i = 1, \dots, r$. If $\lambda, \dots, \lambda_r$ are linearly independent over k, then they are algebraically independent over $\overline{k}$.*

8.2.5. *Example.* Here we consider a single logarithm. Namely we revisit §4.2.2 and the formula in (5.3.1.2),
$$\log_C(\zeta_\theta) = \frac{\pi_q}{\theta}.$$
We have
$$\Phi = \begin{bmatrix} t-\theta & 0 \\ \zeta_\theta^{(-1)}(t-\theta) & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Omega & 0 \\ \Omega L_{\zeta_\theta} & 1 \end{bmatrix}.$$
If we consider the function
$$\Upsilon := tL_{\zeta_\theta} - \zeta_\theta(t-\theta),$$
then
$$\begin{aligned}
\Upsilon^{(-1)} &= tL_{\zeta_\theta}^{(-1)} - \zeta_\theta^{(-1)}(t-\theta^{(-1)}) \\
&= t\left(\zeta_\theta^{(-1)} + \frac{L_{\zeta_\theta}}{t-\theta}\right) - \zeta_\theta^{(-1)}(t-\theta^{(-1)}) \quad \text{(from (5.3.2.1))} \\
&= \frac{tL_{\zeta_\theta}}{t-\theta} - \zeta_\theta \quad (\text{since } \zeta_\theta\theta = -\zeta_\theta^q) \\
&= \frac{\Upsilon}{t-\theta}.
\end{aligned}$$
Therefore,
$$(\Omega\Upsilon)^{(-1)} = \Omega\Upsilon,$$
and so
$$\Upsilon = \frac{c}{\Omega}, \quad c \in \mathbb{F}_q(t).$$
One checks that $c(t) = -1$ identically (by evaluating at $t = \theta$ say), and so
$$\zeta_\theta(t-\theta)\Omega - t\Omega L_\alpha - 1 = 0.$$
Therefore Z_Ψ is defined by the equation,
$$Z_\Psi : \zeta_\theta(t-\theta)X_0 - tX_1 - 1 = 0.$$
It follows that the defining equation for Γ_Ψ is
$$\Gamma_\Psi : tX_1 - X_0 + 1 = 0.$$
Another way to do this is to use the defining equation for Z_Ψ together with (7.1.2.1).

8.3. **Zeta values.** As we saw in §5.4, Carlitz zeta values also fit into this framework. Moreover, we have the following theorem of Chang and Yu.

Theorem 8.3.1 ([Chang-Yu 2007]). *For any positive integer n, the transcendence degree of the field*

$$\overline{k}(\pi_q, \zeta_C(1), \ldots, \zeta_C(n))$$

over $\overline{k}$ is

$$n - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/p(q-1) \rfloor + 1.$$

8.3.2. *Remark.* Although there are many algebraic relations among Carlitz zeta values, this theorem proves that all of them can be accounted for by the relations

$$\begin{aligned} \zeta_C(n) &= r_n \pi_q^n, \quad r_n \in k && \text{(Euler-Carlitz relations)}, \\ \zeta_C(pn) &= \zeta_C(n)^p && \text{(Frobenius p-th power relations)}. \end{aligned}$$

The Euler-Carlitz relations we have already observed in §5.4, and the Frobenius relations are easily seen.

8.3.3. *Short remarks on the proof.* In some respects the proofs of this theorem and of Theorem 8.2.4 are similar, however the situation here is far more complicated. The reason is that zeta values do not appear strictly as Carlitz polylogarithms, but as combinations of them arising through complex generating functions in [Anderson-Thakur 1990]. The problem that arises is that we cannot consider only the zeta values themselves but must include the collection of polylogarithms associated to each one. Ultimately, Chang and Yu develop a procedure for sifting out only the zeta values and arrive at their result.

These methods have subsequently been extended to zeta values with varying constant fields. For $m \geq 1$ and $n \geq 1$, we let

$$\zeta_m(n) = \sum_{\substack{a \in \mathbb{F}_{p^m}[\theta] \\ a \text{ monic}}} \frac{1}{a^n}.$$

Thus $\zeta_m(n)$ is a Carlitz zeta value for $\mathbb{F}_{p^m}[\theta]$, and now we will allow m to vary. The result is that even if we allow m to vary, all relations are still accounted for by the Euler-Carlitz relations and the Frobenius p-th power relations.

Theorem 8.3.4 ([Chang-Papanikolas-Yu]). *For any positive integers s and d, the transcendence degree of the field*

$$\overline{k}\big(\cup_{m=1}^{d} \{\pi_{p^m}, \zeta_m(1), \ldots, \zeta_m(s)\}\big)$$

over $\overline{k}$ is

$$\sum_{m=1}^{d} \Big(s - \lfloor s/p \rfloor - \lfloor s/(p^m - 1) \rfloor + \lfloor s/(p(p^m-1)) \rfloor + 1 \Big).$$

8.4. **Drinfeld modules of rank 2.** Let

$$\rho : \mathbb{F}_q[t] \to \overline{k}[F]$$

be a Drinfeld module of rank 2 over $\overline{k}$. As we saw in §5.5, the periods and quasi-periods of ρ also fit into the Galois group framework. The Galois group associated to ρ satisfies

$$\Gamma_\rho \subseteq \mathrm{GL}_{2/\mathbb{F}_q(t)}\,.$$

Theorem 7.3.1 then implies that

$$\text{tr. deg}\overline{k}(\omega_1, \omega_2, \eta_1, \eta_2) = 4 \quad \Leftrightarrow \quad \Gamma_\rho = \mathrm{GL}_2\,.$$

In recent work Chang and Papanikolas prove this when there is no complex multiplication and $p \neq 2$.

Theorem 8.4.1 ([Chang-Papanikolas])**.** *Let ρ be a Drinfeld module of rank 2 over $\overline{k}$, and let ω_1, ω_2, η_1, η_2 be its associated periods and quasi-periods. If $p \neq 2$ and* $\mathrm{End}(\rho) = \mathbb{F}_q[t]$, *then*

$$\omega_1, \omega_2, \eta_1, \eta_2$$

are algebraically independent over $\overline{k}$.

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