Lecture 2

Frits Beukers

Arithmetic of values of E- and G-function

E-function definition

Definition

An entire function $f(z)$ given by a powerseries

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

with $a_k \in \overline{\mathbb{Q}}$ for all k, is called an E-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.
2. Both $\overline{|a_k|}$ and the common denominator $\mathrm{den}(a_0, \ldots, a_k)$ are bounded by an exponential bound of the form C^k, where $C > 0$ depends only on f.

Siegel-Shidlovskii theorem

Siegel-Shidlovskii, 1929, 1956

Let $(f_1(z), \ldots, f_n(z))$ be a solution vector of a system of first order equations of the form

$$\mathbf{y}'(z) = A(z)\mathbf{y}(z)$$

and suppose that the $f_i(z)$ are E-functions. Let $T(z)$ be the common denominator of the entries of $A(z)$. Let $\alpha \in \overline{\mathbb{Q}}$ and suppose $\alpha T(\alpha) \neq 0$. Then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)) = \mathrm{degtr}_{\mathbb{C}(z)}(f_1(z), f_2(z), \ldots, f_n(z))$$

Generalising Lindemann-Weierstrass

Recall

Lindemann-Weierstrass

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct algebraic numbers. Then

$$e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$$

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Question

Suppose we have $\mathbb{C}(z)$-linear independent E-functions $f_1, f_2, \ldots, f_n$ and α a non-zero algebraic number. Can it be true that the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent?

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Answer: not always

An exceptional set

- Suppose f_1, f_2 are $\mathbb{C}(z)$-linear independent E-functions.
- Consider $g_1 = Af_1 + Bf_2$ and $g_2 = Cf_1 + Df_2$ where $A, B, C, D \in \overline{\mathbb{Q}}[z]$ and $\Delta(z) := AD - BC \not\equiv 0$.
- Then g_1, g_2 are $\mathbb{C}(z)$-linearly independent E-functions.

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- Then g_1, g_2 are $\mathbb{C}(z)$-linearly independent E-functions.
- But for any β with $\Delta(\beta) = 0$ the values $g_1(\beta), g_2(\beta)$ are $\overline{\mathbb{Q}}$-linear dependent.

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Nesterenko-Shidlovskii,1996

Let $f_1, f_2, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations. Then there exists a finite set $S \subset \mathbb{Q}$ such that for any algebraic number α not in S the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent.

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Proof uses the Siegel-Shidlovskii method.

A question by S.Lang

Theorem, FB 2004

Let $f_1(z), \ldots, f_n(z)$ be E-functions which satisfy a system of n first order equations. Let $\alpha \in \overline{\mathbb{Q}}$ and suppose it is not zero or a singularity of the system. Then any $\overline{\mathbb{Q}}$-linear relation between the numbers $f_i(\alpha)$ are specialisation of a $\overline{\mathbb{Q}}(z)$-linear relation between the $f_i(z)$ at $z = \alpha$.

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Corollary

Let $f_1(z), \ldots, f_n(z)$ be E-functions which satisfy a system of n first order equations. Suppose that $f_1(z), \ldots, f_n(z)$ are $\overline{\mathbb{Q}}(z)$-linear independent. Then $f_1(\alpha), \ldots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent.

Relations result from specialisation

Corollary

Let $f_1(z), \ldots, f_n(z)$ be E-functions which satisfy a system of n first order equations. Let $\alpha \in \overline{\mathbb{Q}}$ and suppose it is not zero or a singularity of the system. Let M be a positive integer. Then any $\overline{\mathbb{Q}}$-linear relation between monomials $f_1(\alpha)^{m_1} \cdots f_n(\alpha)^{m_n}$ of degree $\leq M$ is specialisation of a $\overline{\mathbb{Q}}(z)$-linear relation between monomials $f_1(z)^{m_1} \cdots f_n(z)^{m_n}$ of degree $\leq M$ at $z = \alpha$.

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Proof The vector consisting of all monomials $f_1(z)^{m_1} \cdots f_n(z)^{m_n}$ of degree $\leq M$ is a solution vector of a large $K \times K$-system consisting of E-functions with $K = \binom{M+n}{n}$. We apply our Theorem to this vector.

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The proof of the theorem is based on a remarkable theorem by Y.André from 2000.

Remarks on linear differential equations

Consider a linear differential equation

$$q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0$$

where $g_i(z) \in \mathbb{C}[z]$ for all i.

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The zeros of $q_n(z)$ are called the *singularities* of the equation, all other points are called *non-singular*.

Theorem, Cauchy

Suppose $a \in \mathbb{C}$ is a non-singular point. Then the solutions of the equation in $\mathbb{C}[[z-a]]$ form an n-dimensional $\mathbb{C}$-vector space. Furthermore there is a one to one correspondence isomorphism of this space with $\mathbb{C}^n$ given by

$$y(z) \mapsto (y(a), y'(a), y''(a), \ldots, y^{(n-1)}(a)).$$

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Finally, the solutions in $\mathbb{C}[[z-a]]$ all have positive radius of convergence.

Apparent singular points

- It may happen that there exists a basis of solutions in $\mathbb{C}[[z-a]]$ but a is a singularity. In that case we call a an *apparent singularity*.
- For example, if all solutions around $z=a$ have a zero a then a is an apparent singularity.

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- For example, if all solutions around $z = a$ have a zero a then a is an apparent singularity.
  This is because $y \mapsto (y(a), y'(a), \ldots, y^{(n-1)}(a))$ is not bijective any more.

Some more remarks

We abbreviate our equation

$$q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0$$

with $q_i(z) \in \mathbb{C}(z)$ by

$$Ly = 0$$

where $L \in \mathbb{C}(z)[d/dz]$ denotes the corresponding linear differential operator.

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Proposition

Let $Ly = 0$ be a mimimal differential equation for f. Then for any differential equation $L_1 y = 0$ satisfied by f there exists a differential operator L_2 such that $L_1 = L_2 \circ L$.

A miraculous theorem

Theorem, Y.André 2000

Let $f(z)$ be an E-function. Then $f(z)$ satisfies a differential equation of the form

$$z^m y^{(m)} + \sum_{k=0}^{m-1} q_k(z) y^{(k)} = 0$$

where $q_k(z) \in \overline{\mathbb{Q}}[z]$ for all k.

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- The equation from André's theorem need not be the minimal equation of $f(z)$.
- For example, the function $(z-1)e^z$ is an E-function, and its minimal differential equation reads $(z-1)f' = zf$. So we have a singularity at $z = 1$. The equation refered to in André's theorem might be $f'' - 2f' + f = 0$.

A consequence

Corollary, Y.André 2000

Let f be an E-function with *rational* coefficients. Suppose that $f(1) = 0$. Then the minimal differential equation of f has an apparent singularity at $z = 1$.

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The simplest example is again $f = (z-1)e^z$, an E-function which vanishes at $z = 1$. Its minimal differential equation is $(z-1)f' = zf$.

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Then the power series of $f(z)/(1-z)$ reads

$$\frac{f(z)}{1-z} = \sum_{r\geq 0} \frac{g_r}{r!} z^r$$

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Suppose that the common denominator of $f_0, \ldots, f_r$ and the sizes $\overline{|f_r|}$ are bounded by C^r for some $C > 0$. Then clearly the common denominator of $g_0, \ldots, g_r$ are again bounded by C^r.

End of proof

Recall

$$g_r = r! \sum_{k=0}^{r} \frac{f_k}{k!}.$$

To estimate the size of $|g_r|$ we use the fact that $0 = f(1) = \sum_{k\geq 0} f_k/k!$. More precisely,

$$\begin{aligned} |g_r| &= \left| -r! \sum_{k>r} f_k/k! \right| \\ &\leq \sum_{k>r} |f_k|/(k-r)! \\ &\leq \sum_{k>r} C^k/(k-r)! < C^r e^C \end{aligned}$$

So $|g_r|$ is exponentially bounded in r. Hence $f(z)/(1-z)$ is an E-function.

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- This means that the original differential equation for $f(z)$ has a full space of analytic solutions all vanishing at $z=1$.

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- The solutions of the minimal equation of $f(z)$ can be found by multiplying the elements from V by $z-1$.
- This means that the original differential equation for $f(z)$ has a full space of analytic solutions all vanishing at $z=1$.
- So $z=1$ is apparent singularity.

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- This equation has constant coefficients and no singularities.
- Contradiction with $F(1) = 0$ and André Corollary

Generalising André's corollary

Using a combination of André's theorem and some differential galois theory one can prove the following result.

Theorem, FB, 2004

Let $f(z)$ be an E-function and suppose that $f(\alpha) = 0$ for some $\alpha \in \overline{\mathbb{Q}}^*$. Then α is an apparent singularity of the minimal differential equation satisfied by f.

Another proof of Lindemann-Weierstrass

Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers. Suppose there exist $b_1, \ldots, b_n$ not all zero, such that

$$b_1 e^{\alpha_1} + \cdots + b_n e^{\alpha_n} = 0.$$

Let us assume $b_i \neq 0$ for all i.

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- The minimal equation is given by
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which has no singularities.

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- The minimal equation is given by
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which has no singularities.
- We have a contradiction.

In general

Suppose $f_1, \ldots, f_n$ form a solution of an $n \times n$-system consisting of E-functions. Suppose that they are $\mathbb{C}(z)$-linear independent. Let α be a non-zero algebraic number and not a singularity of the system. Suppose that there exist $b_1, \ldots, b_n \in \overline{\mathbb{Q}}$, not all zero, such that

$$b_1 f_1(\alpha) + \cdots + b_n f_n(\alpha) = 0.$$

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- How to contradict this?

In general

Suppose $f_1, \ldots, f_n$ form a solution of an $n \times n$-system consisting of E-functions. Suppose that they are $\mathbb{C}(z)$-linear independent. Let α be a non-zero algebraic number and not a singularity of the system. Suppose that there exist $b_1, \ldots, b_n \in \overline{\mathbb{Q}}$, not all zero, such that

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- Solution: Construct polynomials $P_1, \ldots, P_n$ with the constraints that $P_i(\alpha) = b_i$ for all i and such that the minimal differential equation for the function

$$F(z) = P_1(z) f_1(z) + \cdots + P_n(z) f_n(z)$$

  does not have a singular point at $z = \alpha$.

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- Then $F(z)$ is an E-function vanishing at $z = \alpha$. Therefore its minimal equation should have a singular point at α. But it doesn't by construction.