1 Introduction

The aim of this work is to investigate when the variation of Hodge structures associated to a family of Calabi-Yaus is decomposable. We show that assuming the conjecture "Hodge implies Absolutely Hodge", the Hodge structure being decomposable can be translated into a condition in Galois theory. Finally we look at some numerical data to try and ascertain whether there is sufficient evidence for a conjecture that the variation of Hodge structures is "rarely" decomposable.

1.1 Hodge Theory and Galois Theory

Definition 1.1 *The* total cohomology *of a variety X over an algebraically closed field k is $H^i_{tot}(X) = H^i_{DR}(X) \times \prod_l H^i_{et}(X, \mathbb{Q}_l)$ where H^i_{DR} is the de Rham cohomology and $H^i_{\acute{e}t}$ is the etale cohomology.*

Let $\sigma : k \hookrightarrow \mathbb{C}$ and set $\sigma X = X \times_k \mathrm{Spec}\mathbb{C}$. Recall that we have comparision isomorphisms:

$$H^i_B(\sigma X) \otimes_{\mathbb{Q}} k \overset{\cong}{\to} H^i_{DR}(X)$$

$$H^i_B(\sigma X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \overset{\cong}{\to} H^i_{\acute{e}et}(X, \mathbb{Q}_l)$$

where H^i_B is the Betti cohomology.

Definition 1.2 *Let $\underline{t} = (t_{DR}, (t_l)_l) \in H^{2i}_{tot}(X)$. Then t is a* Hodge cycle *with respect to $\sigma : k \hookrightarrow \mathbb{C}$ if there exists $t_{B,\sigma} \in H^{2i}_B(X)$ such that:*

1. *(rationality) $t_{DR} = comp_{B,DR}(t_{B,\sigma} \otimes 1)$ and $t_l = comp_{B,l}(t_{B,\sigma} \otimes 1)$*
2. *(Hodge condition) $(t_{B,\sigma}) \otimes 1 \in H^{i,i}(\sigma X) \subset H^{2i}_B(\sigma X) \otimes \mathbb{C}$*

Definition 1.3 *We call a Hodge cycle an* absolute Hodge cycle *if it is a Hodge cycle relative to every embedding $\sigma : k \hookrightarrow \mathbb{C}$.*

We will assume the following:

Conjecture 1.4 *Every Hodge cycle is an absolute Hodge cycle.*

Let k_0 be a field (not necessarily algebraically closed) contained in k. Then we have the following commutative diagram:

$$\begin{array}{ccccc} X_{\mathbb{C}} & \longrightarrow & X & \longrightarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k_0) \end{array}$$

There is a natural action of $\mathrm{Gal}(k/k_0)$ on $H^i_{ét}(X, \mathbb{Q}_l)$ and the comparision isomorphisms can be used to get an action on $V = H^i_B(X_{\mathbb{C}})$.

Definition 1.5 *One says that* W *is a* tensor construction *from* V *if it satisfies the following two properties:*
1. $W = V^{\otimes n_1} \otimes \check{V}^{\otimes n_2} \otimes \mathbb{Q}(1)^{\otimes n_3}$
2. $W_{Hdg} = W \bigcap (W \otimes \mathbb{C})^{0,0}$

Using Deligne's article (Art 1, LNM 900, Prop 2.9), the above conjecture implies that the image $W_{Hdg} \hookrightarrow W \otimes \mathbb{Q}_l$ is stabilized by $\mathrm{Gal}(k/k_0)$.

Definition 1.6 *The* Mumford-Tate group $\mathrm{MT}(V)$ *is the largest* $\mathbb{Q}$*-algebraic subgroup of* $GL(V) \times GL(\mathbb{Q}(1))$ *whose rational points fix all the elements of* W_{Hdg}. $MT'(V)$ *is the projection onto* $GL(V)$.

When the associated Hodge structure is polarizable, it can be shown (loc. cit.) that this implies that the Mumford Tate group is reductive. In particular this means that a $\mathbb{Q}$-linear transformation is in $\mathrm{MT}(V)$ iff it stabilizes all Hodge cycles; i.e. this property uniquely characterizes $\mathrm{MT}(V)$.

Proposition 1.7 *The action of* $Gal(k/k_0)$ *normalizes* $MT'(V) \subset GL(V \otimes \mathbb{Q}_l)$.

Proof. Let $\sigma \in \mathrm{Gal}(k/k_0)$, $m \in \mathrm{MT}'$, and $w \in W_{Hdg}$. Then $\sigma w = \acute{w}$ for some $\acute{w} \in W_{Hdg}$. We also observe that $m\acute{w} = \acute{w}$. Then $(\sigma^{-1} m \sigma) w = w$ By the fact mentioned above (i.e. stabilizing all Hodge cycles implies membership of $\mathrm{MT}(V)$) we have that $(\sigma^{-1} m \sigma) \in \mathrm{MT}'(V)$. ▌

1.2 The Mumford-Tate Group of a certain 3-dimensional Hodge structure

In this section we consider a Hodge structure V of weight 3 and rank 4 such that $V \otimes \mathbb{C} = V^{0,3} \oplus V^{1,2} \oplus V^{2,1} \oplus V^{3,0}$ with Hodge numbers all equal to 1. We will assume $V = V_1 \oplus V_2$, $V_1 \otimes \mathbb{C} = V^{0,3} \oplus V^{3,0}$,and $V_2 \otimes \mathbb{C} = V^{1,2} \oplus V^{2,1}$. Working over $\mathbb{C}$, we can pick a basis for $V_1 \otimes \mathbb{C}$ of the form $\left(\begin{smallmatrix} 1 \\ \tau_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ \bar{\tau}_1 \end{smallmatrix}\right)$ and similarly $\left(\begin{smallmatrix} 1 \\ \tau_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ \bar{\tau}_2 \end{smallmatrix}\right)$ for $V_2 \otimes \mathbb{C}$. Using the polarization one gets that $Im(\tau_1) < 0$ and $Im(\tau_2) > 0$.
For a Hodge structure W, we define a map $\mu : \mathbb{G}_m \to \mathrm{GL}(W)$ by $\mu(\lambda)(w^{p,q}) = \lambda^{-p} w^{p,q}$ for $w^{p,q} \in W^{p,q}$.

Proposition 1.8 *The Mumford-Tate group of a Hodge structure W is the smallest algebraic subgroup G of $GL(V) \times \mathbb{G}_m$ such that $\mu(\mathbb{G}_m) \subset G_{\mathbb{C}}$.*

Proof. See the article by Deligne referenced above. ▌

For $W = V_1$ with respect to the basis given above we have that $\mu_1(\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda^{-3} \end{smallmatrix}\right)$,and on V_2 with respect to the given basis $\mu_2(\lambda) = \left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-2} \end{smallmatrix}\right)$.
Now with respect to the standard basis $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, we have

$$\mu_1(\lambda) = \frac{1}{\bar{\tau} - \tau} \left(\begin{smallmatrix} \bar{\tau}\lambda^{-3} - \tau & -\lambda^{-3} + 1 \\ \bar{\tau}\lambda^{-3}\tau - \tau\bar{\tau} & -\lambda^{-3}\tau + \bar{\tau} \end{smallmatrix}\right)$$

$$\mu_2(\lambda) = \frac{1}{\bar{\tau} - \tau} \left(\begin{smallmatrix} \bar{\tau}\lambda^{-2} - \tau\lambda^{-1} & -\lambda^{-2} + \lambda^{-1} \\ \bar{\tau}\lambda^{-2}\tau - \lambda^{-1}\tau\bar{\tau} & \lambda^{-2}\tau + \lambda^{-1}\bar{\tau} \end{smallmatrix}\right)$$

and that $\det\mu_1(\lambda) = \det\mu_2(\lambda) = \lambda^{-3}$.

Now, using the calculations for λ, we can remove most of the possibilities for $\mathrm{MT}'(V_i)$. The only possibilities are $\mathrm{MT}'(V_i) = \mathrm{GL}_2$ or T^2.
Now we figure out what the possibilities for $\mathrm{MT}'(V)$ are for dimension 4. One can show that $\mathrm{MT}'(V) \subset \mathrm{MT}'(V_1) \oplus \mathrm{MT}'(V_2)$. Hence, $\mathrm{MT}'(V) \subset G$ where G is one of the following:

a. $\mathrm{GL}_2 \times \mathrm{GL}_2$

b. $\mathrm{GL}_2 \times T^2$
c. $T^2 \times \mathrm{GL}_2$
d. $T^2 \times T^2$

Proposition 1.9 *$MT'(V)$ is one of the following groups:*
a. $\{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) | g_i \in GL_2, detg_1 = detg_2\}$
b. $\{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) | g_1 \in GL_2, g_2 \in T^2, detg_1 = detg_2\}^o$
c. $\{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) | g_1 \in T^2, g_2 \in GL_2, detg_1 = detg_2\}^o$
d. $\{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) | g_1 \in T^2, g_2 \in T^2, detg_1 = detg_2\}$
e. $\{\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} | t_1 t_2 = t_3 t_4 \ and \ \frac{t_1}{t_2} = (\frac{t_3}{t_4})^3\} = MT'(V)_{\mathbb{C}}$

Proof. First note that the equality of determinants is an immediate consequence of our calculations following Proposition 1.8. In case a one can see that the Mumford-Tate group is either the one indicated in the statement of the theorem or it's a subgroup thereof with an additional defining condition $g_1 = g_2$ or $g_2 = g_0 g_1 g_0^{-1}$ for some 2×2 invertible matrix g_0. Suppose the first possibility occurs, i.e. $g_1 = g_2 = g$. Then the matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ commutes with all the elements of $\mathrm{MT}'(V)$. Hence the corresponding element in $(\check{V} \otimes V)_{\mathbb{C}}$ is fixed by the complexification of the Mumford-Tate group. It follows that $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ corresponds to an element of bidegree (0,0) in the weight 0 Hodge structure $(\check{V} \otimes V)_{\mathbb{Q}}$. Thus it gives an endomorphism of Hodge structures of V, which maps V_1 isomorphically onto V_2, which is absurd as those Hodge structures are not isomorphic. This rules out the case $g_1 = g_2$. The argument that allows one to exclude the second case is similar.

For case b note that $\mathrm{MT}'(V)$ contains both SL_2 and T which intersect trivially, hence the claim follows from a dimension argument.

For d and e we have $2 \leq \dim \mathrm{MT}'(V) \leq 3$. If $\dim \mathrm{MT}'(V) = 3$ we obtain case d. If $\dim \mathrm{MT}'(V) = 2$, the two tori are isogenous and the associated quadratic imaginary fields are the same. Consideration of the action of the Galois group forces the additional relation occuring in case e. ∎

Observation 1.10 *In cases b-e in the above proposition, an element of the normalizer of $MT'(V)$ in $GL(V)$ doesn't interchange the pieces of V, but in case a it can happen. However, the matrix that flips the pieces has trace 0.*

1.3 A Criterion for Irreducibility

Now we consider the family Y_s of hypersurfaces in $\mathbb{P}^4$ given by:

$$Y_s : s(y_0 + y_1 + y_2 + y_3 + y_4)^5 = y_0 y_1 y_2 y_3 y_4.$$

We want to apply the results of the previous sections to determine the $s \in \mathbb{C}$ for which $H^3(Y_s, \mathbb{Q})$ is reducible. We will give a sufficient condition for the irreducibility of the Hodge structure.
Assume the Hodge structure is reducible for some s. Then from de Jong's lecture we know that $s \in \bar{\mathbb{Q}}$ if we assume Conjecture 1.4. Let $k_0 = \mathbb{Q}[s]$, and let $\mathcal{O}_{k_0}$ be the ring of integers of k_0. Pick a prime $\wp \subset \mathcal{O}_{k_0}$ and let $Frob_\wp$ be a Frobenius element in $\mathrm{Gal}(\bar{\mathbb{Q}}/k_0)$ for $\wp$. For simplicity, assume that $\wp$ is a degree one prime. Also assume that $\wp$ lies above $(p) \subset \mathbb{Z}$. We can then compute the characteristic polynomial $P_\wp$ of $Frob_\wp$ acting on $H^3_{et}(Y_s \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l) \overset{\cong}{\to} H^3_B(Y_s \otimes \mathbb{C}, \mathbb{Q}) \otimes \mathbb{Q}_l$ for $l \neq p$. (Note that $P_\wp$ depends on s, but for simplicity we omit the s here.) We can compute $P_\wp$ by reducing the equation for Y_s mod $\wp$ and by counting points over $\mathbb{F}_p, \mathbb{F}_{p^2}$. (We know the contributions of the pieces $H^i_{et}(Y_s \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l)$ for $i \neq 3$, because the Betti numbers for those pieces are all 0 or 1.) Then there are two cases to consider:
Case 1: $Frob_\wp$ flips the two factors V_1 and V_2. Then, as observed above (See Observation 1.10), we know that $\mathrm{trace}(Frob_\wp) = 0$. This occured only once in our numerical data, so we will ignore this case.
Case 2: $Frob_\wp$ does not flip the pieces. In this case, the characteristic polynomial $P_\wp$ which has degree 4 must factor into two quadratic factors over $\mathbb{Q}_l$ for every $l \neq p$, because $Frob_\wp$ preserves the two two-dimensional spaces V_1 and V_2. We know the shape of the polynomial $P_\wp$:

$$P_\wp = x^4 + a_1 x^3 + a_2 x^2 + p^3 a_1 x + p^6$$

and the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ can be numbered so that they satisfy the equations $\lambda_1 \lambda_2 = p^3$, $\lambda_3 \lambda_4 = p^3$.
Now we can ask: When is there an l such that $P_\wp$ is irreducible over $\mathbb{Q}_l$? It is certainly sufficient that

1. $P_\wp$ is irreducible over $\mathbb{Q}$ and
2. If K is the splitting field of $P_\wp$, then $\mathrm{Gal}(K/\mathbb{Q})$ contains a 4-cycle.

However, because of the shape of the polynomial and relations on the roots, the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ can only be the Klein 4-group V_4 or it contains a 4-cycle. Therefore, the group we wish to eliminate is V_4. We can check if $\mathrm{Gal}(K/\mathbb{Q})$ is V_4 by checking whether the discriminant of $P_\wp$ is a square in $\mathbb{Q}^*$.

With notation as above, we can summarize our results as follows:

Summary 1.11 *Given an s in $\bar{\mathbb{Q}}$, if there exists a prime $\wp$ of $\mathcal{O}_{k_0}$ such that $P_\wp$ satisfies:*

1. *trace*$(P_\wp) \neq 0$
2. *$P_\wp$ is irreducible over $\mathbb{Q}$*
3. *disc*$(P_\wp)$ *is not a square in $\mathbb{Q}^*$*

then $H^3(Y_s, \mathbb{Q})$ is irreducible.

1.4 Some Numerical Data

Now we present some numerical data to see if the $P_\wp$ are ever reducible for any s. There certainly is not enough evidence to even conjecture whether the Hodge structure is ever decomposable or not because we were only able to count the points of Y_s over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ for very small primes p. We are excluding the case where $p = 5$, and for each p we look at $s \in \{1, \cdots, p-1\}$, $s \neq 1/5^5 \mod p$. In the following we compute the characteristic polynomial $P_\wp = P_{\wp,s}$ for a given prime p and all allowed values of s.

s	$P_\wp$ for $p = 3$	irredcible ?	disc a square ?
1	$x^4 + 5x^3 + 45x^2 + 135x + 729$	yes	no

s	$P_\wp$ for $p = 7$	irreducible ?	disc a square ?
1	$x^4 + 5x^3 + 385x^2 + 7^3 \cdot 5x + 7^6$	yes	no
2	$x^4 + 25x^3 + 350x^2 + 7^3 \cdot 25x + 7^6$	yes	no
3	$x^4 + 10x^3 + 420x^2 + 7^3 \cdot 10x + 7^6$	yes	no
4	$x^4 - 5x^3 - 210x^2 - 7^3 \cdot 5x + 7^6$	yes	no
6	$x^4 - 35x^3 + 805x^2 - 35 \cdot 7^3x + 7^6$	yes	no

For $p = 11$, again, for each allowed value of s, the corresponding characteristic polynomial has nonzero trace, is irreducible and its discriminant is not a square in $\mathbb{Q}^*$.

s	$P_\wp$ for $p = 13$	irreducible ?	disc a square ?
4	$x^4 + 10x^3 - 910x^2 + 13^3 \cdot 10x + 13^6$	**no** !	no
6	$x^4 - 120x^3 + 7670x^2 + 13^3 \cdot (-120)x + 13^6$	**no** !	no

All other allowed values for s are ok when p is 13.
For $p = 17$ all allowed values of s are ok.

s	$P_\wp$ for $p = 19$	irreducible ?	disc a square ?
13	$(x^2 - 95x + 19^3)(x^2 + 100x + 19^3)$	**no** !	no

All other allowed values for s are ok when s is 19.
When p is 23 and $s = 18$, the corresponding characteristic polynomial is $x^4 + 8050x^2 + 23^6$ which has trace 0 and whose discriminant is a square.
When $p = 31$ and $s = 5$, the corresponding charcateristic polynomial factors as $(x^2 - 217x + 31^3) \cdot (x^2 + 108x + 31^3)$.
When $p = 41$ and $s = 18$, the corresponding characteristic polynomial factors as $(x^2 - 372x + 41^3) \cdot (x^2 + 328x + 41^3)$.