CHOW GROUPS CAN BE FINITE DIMENSIONAL, IN SOME SENSE

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ABSTRACT. Let S be a surface with $p_g(S) > 0$. In 1969, Mumford proved that its Chow group CH_*S is not "finite dimensional" in some sense. In this paper, we propose another definition of "finite dimensionality" for the Chow groups. Using this new definition, at least the Chow group of some surface S with $p_g(S) > 0$ (for example, the product of two curves) becomes finite dimensional.

The finite dimensionality of the Chow group follows from the finite dimensionality of the Chow motive. It turns out that the finite dimensionality of the Chow motive is a very strong property. For example, we can prove a generalized Bloch's conjecture (representability of Chow groups of surfaces with $p_g(S) = 0$) under the assumption of the finite dimensionality of the Chow motive of the surface S.

CONTENTS

1. INTRODUCTION

For a complete variety X, the Chow group of X is the group of algebraic cycles, identifying two cycles if they are in the same family parametrized by $\mathbb{P}^1$. In this paper, we consider the Chow groups with rational coefficients, namly we will write simply $CH_*(X)$ for $CH_*(X) \otimes \mathbb{Q}$. We will denote by $CH_*(X)_{\mathbb{Z}}$ when we mean the Chow group with integral coefficients. When C is a smooth algebraic curve, the continuous part of $CH_0(X)_{\mathbb{Z}}$ is represented by the Jacobian variety of C. Because one can parametrize all the effective algebraic cycles by (a limit of) the Chow varieties, it is natural to expect that the Chow group is always represented by some variety. When S is an algebraic surface, Mumford [14] proved that if $p_g(S) > 0$, then $CH_0(S)$ is "infinite dimensional" in the following sense: Let W be a subvariety of $Sym^n(S)$ (the n-th symmetric product variety of S) such that by the natural map $Sym^n(S) \to CH_0(S)$, W is sent to one point, then Mumford proved that $\dim(W) \leq n$. This result suggests that if $CH_0(S)$ is represented by some variety, then the dimension of the variety is larger than n (for any n), hence the naming "infinite dimensional."

The Chow group $CH_0(S)$ may not be representable, but still we may hope that it is "finite dimensional" from a different viewpoint. The goal of this paper is to show that if we accept to use another definition of finite dimensionality, then at least the Chow groups of the products of curves and their quotients are finite dimensional.

For example, let us take curves C and D, smooth nonrational curves, and let X be the product $C \times D$. Then $p_g(X)$ is the product of the genera of the curves C and D, hence $CH_0(X)$ is infinite dimensional in the sense of Mumford. To study the structure of $CH_0(X)$, let us look at the albanese map $alb : CH_0(X) \to \mathbb{Z} \oplus Alb(X)$. Because $\mathbb{Z} \oplus Alb(X)$ is finite dimensional in the sense of Mumford, we should have a nonzero kernel for the albanese map, which is Mumford-"infinite dimensional", and the kernel is called the "albanese kernel." We will show that the albanese kernel looks like "finite dimensional" in the following sense:

Definition 1.1. *For algebraic cycles $c_1, \cdots, c_n \in CH_*(X)$, define their wedge product by*

$$c_1 \wedge \cdots \wedge c_n = \sum_{\sigma \in \mathfrak{S}_n} \frac{sgn(\sigma)}{n!} \cdot c_{\sigma(1)} \times \cdots \times c_{\sigma(n)} \in CH_*(\overbrace{X \times \cdots \times X}^{n-\text{times}})$$

where $c_{\sigma(1)} \times \cdots \times c_{\sigma(n)}$ is the exterior product. (See the Chapter 1 of [5] *for the definition of the exterior products.)*

Theorem 1.2. *(Corollary 5.14) Assume that the base field is algebraically closed. Let $c_1, \cdots, c_n$ be cycles in the albanese kernel in $CH_0(X)$, where X is a product of two curves. When n is larger than $4p_g(X)$, then $c_1 \wedge \cdots \wedge c_n = 0$.*

So we can pick only finitely many "linearly independent" cycles from the albanese kernel, hence it looks like finite dimensional.

It is interesting to see that the continuous part of $CH_0(C)$ of a curve C does not look like finite dimensional in the sense above, namely we have the following result.

Proposition 1.3. *(Theorem 8.5) Let C be a smooth projective curve. For any $n > 0$, there exists 0-cycles $c_1, \cdots, c_n$ in $CH_*(C)$ such that c_i's have degree 0 and $c_1 \wedge \cdots \wedge c_n \neq 0$.*

But we know that $CH_0(C)_{\mathbb{Z}}$ is represented by a finite dimensional variety, and it is finite dimensional in the following sense:

Definition 1.4. *For algebraic cycles $c_1, \cdots, c_n \in CH_*(X)$, we define their* symmetric product *$Sym(c_1, \cdots, c_n)$ by*

$$Sym(c_1, \cdots, c_n) = \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{n!} \cdot c_{\sigma(1)} \times \cdots \times c_{\sigma(n)}.$$

Theorem 1.5. *(Corollary 4.3) Let C be a smooth curve, and $n > 2g(C)$. If $c_1, \cdots, c_n \in CH_0(C)$ are 0-cycles with degree 0, then $Sym(c_1, \cdots, c_n) = 0$.*

So the Chow group of curves is "finite dimensional" in a different way.

To study the Chow group of an algebraic curve, the standard geometric idea is to construct the symmetric product of the curve and look at the morphism to the

Jacobian variety. Mumford's result suggests that the same idea would not work for surfaces with $p_g > 0$. Our result suggests that some kind of "wedge product" of surfaces would help us to understand the Chow group of a surface, but the author has no idea what a wedge product should be.

Theorems 1.2 and 1.5 follow from the finite dimensionality of Chow motives. For example, we will construct a Chow motive $h^1(C)$ for a curve C, and observe that $\{$0-cycles with degree 0 in $CH_0C\} = CH_*(h^1(C))$. Then computation shows that the symmetric product motive satisfies $Sym^n(h^1(C)) = 0$ for $n > 2g(C)$, which is the "finite dimensionality" of $h^1(C)$, and from which Theorem 1.5 follows, because we will have $sym(c_1, \cdots, c_n) \in CH_*(Sym^n(h^1(C)))$. In order to prove the finite dimensionality of the albanese kernel, we have to check: i) Because the Chow motives $h^1(C)$ and $h^1(D)$ are finite dimensional, their tensor product $h^1(C)\otimes h^1(D)$ is also finite dimensional, in the sense $\wedge^n(h^1(C) \otimes h^1(D)) = 0$ for $n > 4g(C)g(D)$. ii) $\{$The albanese kernel in $CH_*(C \times D)\} = CH_*(h^1(C) \times h^1(D))$.

The finite dimensionality of Chow motives turns out to be surprisingly strong condition. For example, if a Chow motive M is finite dimensional, then its dimension is same as the dimension of the cohomology:

$$\dim M = \dim_{\mathbb{Q}} H^*(M, \mathbb{Q}).$$

In particular, if a finite dimensional motive M is homologically trivial, i.e., $H^*(M, \mathbb{Q}) = 0$, then we have $M = 0$. It means that we can detect such a strong information at the level of rational equivalence from the information of homological equivalenece. As a result, assuming the finite dimensionality of Chow motives, we obtain a generalized Bloch's conjecture (or the inverse of Jannsen's theorem) which says that if the cycle map is surjective, then it is injective. So for quotients of the products of curves, we obtain the Bloch's conjecture.

The author would like to thank Amnon Neeman, William Fulton, Spencer Bloch and C. Soule for their encouragement and mathematical help. Also he is grateful to Vladimir Voevodsky, Angelo Vistoli, Ichiro Shimada and Alexei Davydov for helpful discussions. He appreciates the hospitality and the support of University of Utah, University of Virginia, Max-Planck-Institut für Mathematik and Tata Institute of Fundamental Research while this research was conducted and this manuscript was prepared.

We work in the category of algebraic schemes over a fixed field κ. When X is an algebraic scheme, CH_*X is the Chow group of X, with $\mathbb{Q}$-coefficients. The Chow group of a variety X with integral coefficients is denoted by $CH_*X_{\mathbb{Z}}$. We denote a point $Spec(\kappa)$ by Pt.

2. Category of Chow Motives

In this section we construct the category of Chow motives and study its basic properties.

Definition 2.1. *When X and Y are smooth complete schemes (or more generally complete Alexander schemes, for example, varieties with only quotient singularities [19]), we define* the correspondence *from X to Y to be an element of $CH_*(X \times Y)$.*

When $\alpha \in CH_*(X \times Y)$ is a correspondence from X to Y, we write $\alpha : X \vdash Y$. When $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$ are correspondences, we define their composition $\beta \circ \alpha$ by

$$\beta \circ \alpha = \pi_{XZ*}(\pi^*_{XY}\alpha \cdot \pi^*_{YZ}\beta)$$

where π's are projections from $X \times Y \times Z$, and "$\cdot$" is the intersection product.

When $f : Y \to X$ is a morphism of schemes, then we define $\overline{f} : X \vdash Y$ to be $\iota^*[\Gamma_f]$ where $\iota : X \times Y \to Y \times X$ is the involution $\iota(x, y) = (y, x)$ and Γ_f is the graph of the morphism f.

Remark 2.2. When $f : Y \to X$ and $g : Z \to Y$ are morphisms, we have $\overline{f \circ g} = \overline{f} \circ \overline{g}$.

Definition 2.3. *Let us define the category of* Chow motives.

The objects are the pairs (X, α) where X is a smooth complete scheme and $\alpha : X \vdash X$ is a correspondence such that $\alpha \circ \alpha = \alpha$.

Morphisms from (X, α) to (Y, β) are defined by

$$Hom\big((X, \alpha), (Y, \beta)\big) = \{\gamma : X \vdash Y | \gamma \circ \alpha = \gamma = \beta \circ \gamma\}.$$

The identity morphism of the motive (X, α) is defined by $id_{(X,\alpha)} = \alpha : X \vdash X$. We write $[\Delta_M]$ for $\alpha \in CH_*(X \times X)$.

Definition 2.4. *When X is a smooth complete scheme, and $id_X : X \vdash X$ be the identity morphism, then $(X, \overline{id_X})$ determines a Chow motive. We call it* the motive of X*, and denote it by $\overline{X}$.*

Remark 2.5. In Definition 2.3, we don't assume any dimension condition for morphisms between Chow motives. Except for this, our definition coincides with Example 16.1.12 of [5]. Our definition $X \to \overline{X}$ and $f \to \overline{f}$ determines a contravariant functor from the category of smooth complete algebraic varieties to the category of Chow motives. A morphism from (X, α) to (Y, β) is represented by an algebraic cycle $\gamma \in CH_*(X \times Y)$. The category of Chow motives is nothing but the category of finitely generated projective bivariant sheaves in [8].

Definition 2.6. *When $M = (X, \alpha)$ is a motive, then we define its* Chow group *$CH_*(M)$ to be $\alpha_*(CH_*(X)) \subset CH_*(X)$. Also we define its* homology group *$H_*(M)$ to be $\alpha_*(H_*(X)) \subset H_*(X)$. It is easy to see that morphisms of motives acts functorially on the Chow groups and homology groups, hence they are well-defined up to isomorphisms. As the cycle maps of varieties $cl : CH_*(X) \to H_*(X)$ commute with correspondences,* the cycle map of a motive *$cl : CH_*(M) \to H_*(M)$ is also well-defined.*

Definition 2.7. *When $M = (X, \alpha)$ is a motive, we call M to be* homologically equivalent to zero *if the algebraic cycle α is homologically equivalent to zero.*

When $f : (X, \alpha) \to (Y, \beta)$ is a morphism of motives represented by $\gamma \in CH_*(X \times Y)$, we call f to be homologically equivalent to zero *if γ is homologically equivalent to zero, and we call f to be* algebraically equivalent to zero *if γ is algebraically equivalent to zero.*

Remark 2.8. We can also define algebraically equivalent to zero motives in a similar way, but by the theorem of Voevodsky [18], such motives are zero. More generally, it is expected that homologically equivalent to zero motives are zero, which we will prove for finite dimensional motives in Corollary 7.3.

Definition 2.9. *Let (X, α) and (Y, β) be Chow motives.*

i) Define their direct sum by $(X, \alpha) \oplus (Y, \beta) = (X \coprod Y, (\alpha, \beta))$ where

$$(\alpha, \beta) \in CH_*(X \times X) \oplus CH_*(Y \times Y) \subset CH_*\big((X \coprod Y) \times (X \coprod Y)\big).$$

ii) Define the tensor product of (X,α) *and* (Y,β) *by* $(X,\alpha)\otimes(Y,\beta) = (X\times Y, \pi_X^*\alpha\cdot\pi_Y^*\beta)$ *where* $\pi_X : (X\times Y)\times(X\times Y)\to X\times X$ *and* $\pi_Y : (X\times Y)\times(X\times Y)\to Y\times Y$ *are projections. When we take the tensor product of a motive* M *with itself many times, we denote it by*

$$\overbrace{M\otimes\cdots\otimes M}^{n \text{ times}} = M^{\otimes n}.$$

Remark 2.10. We can consider $[\Delta_M]$ as an element of $CH_*(M\otimes M)$.

Proposition 2.11. *Chow motive* (X,α) *is isomorphic to the motive of a point if and only if* α *can be written as* $\alpha = \pi_1^*c\cdot\pi_2^*d$ *where* $\pi_i : X\times X\to X$ *are the i-th projections* $(i = 1,2)$*, and* c *and* d *are elements of* $CH_*(X)$*. In this case, the cycles* c *and* d *satisfy* $deg(c\cdot d) = 1$.

Proof: When we have a correspondence $\alpha = \pi_1^*c\cdot\pi_2^*d$, then direct calculation shows that $\alpha\circ\alpha = deg(c\cdot d)\cdot\alpha$. So in order that α is a non-zero idempotent and hence (X,α) is a Chow motive, we should have $deg(c\cdot d) = 1$.

When $\varphi : (X,\alpha)\to\overline{Pt}$ is an isomorphism, we have $\varphi\in CH_*(X\times Pt)$ with $\varphi\circ\alpha = \varphi$. Let $\psi : \overline{Pt}\to(X,\alpha)$ be its inverse, then we have $\psi\in CH_*(Pt\times X)$. Now we have

$$\alpha = id_{(X,\alpha)} = \psi\circ\varphi = \pi_{X\times X*}(\pi_{X\times Pt}^*\varphi\cdot\pi_{Pt\times X}^*\psi) = \pi_1^*\varphi\cdot\pi_2^*\psi$$

So letting $c = \varphi$ and $d = \psi$, we have $\alpha = \pi_1^*c\cdot\pi_2^*d$.

Conversely assume that we have a Chow motive (X,α) with $\alpha = \pi_1^*c\cdot\pi_2^*d$, then the degree of $c\cdot d$ must be 1. Considering $c : X\vdash Pt$ and $d : Pt\vdash X$ as correspondences, we calculate $c\circ\alpha = deg(c\cdot d)\cdot c$ and $\alpha\circ d = deg(c\cdot d)\cdot d$, so they determine the morphism of motives $c : (X,\alpha)\to\overline{Pt}$ and $d : \overline{Pt}\to(X,\alpha)$. Now direct calculation shows that $c\circ d = id_{\overline{Pt}}$ and $d\circ c = id_{(X,\alpha)}$. □

Definition 2.12. *Let* $M_i = (X_i,\alpha_i)$ *and* $N_i = (Y_i,\beta_i)$ *be motives and* $f_i : M_i\to N_i$ *be morphisms of motives* $(i = 1,2,\dots,k)$*. Represent* f_i *by* $\gamma_i\in CH_*(X_i\times Y_i)$*, and we define the tensor product of* f_i*'s by*

$$\bigotimes_{i=1}^n f_i = \pi_1^*\gamma_1\cdots\pi_n^*\gamma_n\in CH_*(\prod_{i=1}^n X_i\times\prod_{i=1}^n Y_i)$$

where $\pi_i : \prod_{j=1}^n X_j\times\prod_{j=1}^n Y_j\to X_i\times Y_i$ *are the projections. When* $f = f_1 = f_2 = \cdots = f_n$*, we denote the tensor product* $\otimes_{i=1}^n f$ *by* $f^{\otimes n}$

Definition 2.13. *A morphism between motives* $f : M\to N$ *is called* smash nilpotent *if* f *satisfies* $f^{\otimes n} = 0$ *for some* $n > 0$.

Remark 2.14. Voevodsky proved that when f is algebraically equivalent to zero, then f is smash nilpotent [18]. Smash nilpotent endomorphisms turn out to be nilpotent, and we will prove a stronger nilpotency in Proposition 2.16.

Proposition 2.15. *Let* $f,g : M\to N$ *be smash nilpotent morphisms between motives. Then the morphism* $f+g : M\to N$ *is also smash nilpotent.*

Proof: Let us say $f^{\otimes n} = 0$ and $g^{\otimes m} = 0$. Then one can expand $(f+g)^{\otimes(n+m-1)}$ to see that each monomial contains at least n-times of the factor f or at least m-times of the factor g, so it is 0. □

Proposition 2.16. *Let $f : M \to N$ be a smash nilpotent morphism, say $f^{\otimes n} = 0$, and $g_1, \dots, g_{n-1} : N \to M$ be morphisms of motives. Then we have*

$$f \circ g_{n-1} \circ f \circ g_{n-2} \circ \cdots \circ g_1 \circ f = 0 : M \to N.$$

In particular when $f : M \to M$ is smash nilpotent, then f is nilpotent.

Proof: Let us represent the motives M and N by $M = (X, \alpha)$ and $N = (Y, \beta)$, then f is represented by an algebraic cycle, say $\gamma_f \in CH_*(X \times Y)$, and g_i by $\gamma_i \in CH_*(Y \times X)$. Take projections $\pi_i : X \times Y \times X \times Y \times \cdots X \times Y \to X \times Y$ defined by $\pi_i(x_1, y_1, \dots, x_n, y_n) = (x_i, y_i)$, and $p_i : X \times Y \times X \times Y \times \cdots X \times Y \to Y \times X$ defined by $p_i(x_1, y_1, \dots, x_n, y_n) = (y_i, x_{i+1})$. Also define $P : X \times Y \times X \times Y \times \cdots X \times Y \to X \times Y$ by $P(x_1, y_1, \dots, x_n, y_n) = (x_1, y_n)$. Then by definition, the composition $f \circ g_{n-1} \circ \cdots \circ g_1 \circ f$ is represented by

$$P_*(\pi_1^*\gamma_f \cdot p_1^*\gamma_1 \cdot \pi_2^*\gamma_f \cdot p_2^*\gamma_2 \cdots p_{n-1}^*\gamma_{n-1} \cdot \pi_n^*\gamma_f).$$

But because $f^{\otimes n}$ is represented by $\pi_1^*\gamma_f \cdots \pi_n^*\gamma_f$, which is 0 by assumption, we have $f \circ g_{n-1} \circ \cdots \circ g_1 \circ f = 0$. For the nilpotency of f, apply $g_i = id_M$. ☐

3. Finite dimensionality of motives

Let M be a motive, and consider $M^{\otimes n}$ as in Definition 2.9. In this section, we will study the decomposition of $M^{\otimes n}$ by the natural action of $\mathfrak{S}_n$ on $M^{\otimes n}$. In particular we will define the motives $Sym^n M$ and $\wedge^n M$, and use them to define the notion of finite dimensionality of motives, the main tool of this paper.

Let us start by reviewing the representation of $\mathfrak{S}_n$ to fix the notation.

Proposition-Definition 3.1. *Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ be a partition of n, i.e., $\sum_{i=1}^k \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, where λ_i's are integers. Each such λ determines an irreducible representation W_λ of $\mathfrak{S}_n$ on $\mathbb{Q}$. Define $\psi_\lambda \in \mathbb{Q}\mathfrak{S}_n$ to be $(\dim W_\lambda)/n! \cdot \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_{W_\lambda}(\sigma)} \cdot \sigma$, where $\chi_{W_\lambda} : \mathfrak{S}_n \to \mathbb{Z}$ is the character of W_λ then ψ_λ is the projector to the irreducible representations isomorphic to W_λ's, and we have $\sum \psi_\lambda = 1$ where λ runs over all partitions of n. Also we have $\psi_\lambda \cdot \psi_\lambda = \psi_\lambda$, and $\psi_\lambda \cdot \psi_\mu = 0$ when $\lambda \neq \mu$.*

Proof: See [6]. ☐

Proposition-Definition 3.2. *Let X be a variety, and define X^n to be $X \times X \times \cdots \times X$ (n times), and define the action of $\mathfrak{S}_n$ on X^n by $\sigma(x_1, \cdots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \cdots, x_{\sigma^{-1}(n)})$. Let $f_\sigma : X^n \to X^n$ be the morphism of the action of σ, and define the correspondence $d_\lambda : X^n \vdash X^n$ to be $(\dim W_\lambda)/n! \cdot \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_{W_\lambda}} \cdot \overline{f_\sigma}$. Then we have $d_\lambda \circ d_\lambda = d_\lambda$, $d_\lambda \circ d_\mu = 0$ for $\lambda \neq \mu$ and $\Sigma d_\lambda = \overline{id_X}$. In particular, the motive of X^n decomposes into submotives $\oplus(X, d_\lambda)$. We denote the motive (X, d_λ) by $\mathbb{T}_\lambda X$.*

Proof: By Remark 2.2, we have $\overline{f_\sigma} \circ \overline{f_\tau} = \overline{f_\sigma \circ f_\tau} = \overline{f_{\sigma\tau}}$. So the multiplications of d_λ's coincide with the multiplications of ψ_λ's in the group ring $\mathbb{Q}\mathfrak{S}_n$. ☐

Definition 3.3. *Let $M = (X, \alpha)$ be a motive. Take the projections $\pi_i : X^n \times X^n \to X \times X$ defined by $\pi_i(x_1, \cdots, x_n, y_1, \cdots, y_n) = (x_i, y_i)$. Let us define the correspondence $\alpha^{(n)} : X^n \vdash X^n$ to be $\pi_1^*\alpha \cdot \pi_2^*\alpha \cdots \pi_n^*\alpha$. We define the motive M^n to be $(X^n, \alpha^{(n)})$.*

Lemma 3.4. *The correspondences $\overline{f_\sigma}$ and $\alpha^{(n)}$ commute each other. Also the correspondences d_λ and $\alpha^{(n)}$ commute each other, namely we have $d_\lambda \circ \alpha^{(n)} = \alpha^{(n)} \circ d_\lambda$. In particular, $\alpha^{(n)} \circ d_\lambda$ is idempotent*

Proof: It is enough to show that the correspondence f_σ commutes with $\alpha^{(n)}$. Let $\pi_{i,\sigma}$ be the projection defined by $\pi_{i,\sigma}(x_1, \cdots, x_n, y_1, \cdots, y_n) = (x_i, y_{\sigma^{-1}(i)})$. Then direct calculation shows that both correspondences $\overline{f_\sigma} \circ \alpha^{(n)}$ and $\alpha^{(n)} \circ \overline{f_\sigma}$ coincide with $\Pi_{i=1}^n \pi_{i,\sigma}^* \alpha$. □

Definition 3.5. *When $M = (X, \alpha)$ is a motive, then $(X^n, d_\lambda \circ \alpha^{(n)})$ is also a motive by Lemma 3.4. We denote this motive by $\mathbb{T}_\lambda M$. When $\lambda = (n)$, we denote $\mathbb{T}_{(n)} M$ by $Sym^n M$, and call it the n-th symmetric product of M. When $\lambda = (1, 1, \cdots, 1)$, we denote $\mathbb{T}_{(1,1,\cdots,1)} M$ by $\wedge^n M$, and call it the n-th wedge product of M.*

Remark 3.6. When $M = (X, \overline{id_X})$ is the motive of a variety X, then $Sym^n M$ is isomorphic to the motive of $Sym^n X$, the n-th symmetric product variety of X.

Definition 3.7. *Let M be a motive. The motive M is* evenly finite dimensional *if there exists a positive integer n such that $\wedge^n M = 0$.*

The motive M is oddly finite dimensional *if there exists a positive integer n such that $Sym^n M = 0$.*

The motive M is finite dimensional *if M can be written as a direct sum $M = M^+ \oplus M^-$ where M^+ is evenly finite dimensional and M^- is oddly finite dimensional.*

Definition 3.8. *When M is evenly finite dimensional motive, then* the dimension of *M is the maximal integer n such that $\wedge^n M \neq 0$. When M is oddly finite dimensional motive, then* the dimension of *M is the maximal integer n such that $Sym^n M \neq 0$. Later in Definition 6.4, we will define the dimension of finite dimensional motive $M = M^+ \oplus M^-$ to be* $\dim(M^+) + \dim(M^-)$.

Proposition 3.9. *If a motive M is evenly finite dimensional, then $H_i(M, \mathbb{Q}) = 0$ when i is odd. If M is oddly finite dimensional, then $H_i(M, \mathbb{Q}) = 0$ when i is even. For both cases, we have* $\dim M \geq \dim_{\mathbb{Q}} H_*(M, \mathbb{Q})$.

Proof: Decompose $H_*(M, \mathbb{Q}) = H_{even}(M, \mathbb{Q}) \oplus H_{odd}(M, \mathbb{Q})$, where, for example, $H_{odd}(M, \mathbb{Q}) = \bigoplus H_{2k+1}(M, \mathbb{Q})$. Then it is easy to see that

$$H_*(Sym^n M, \mathbb{Q}) = \bigoplus_{i+j=n} Sym^i H_{even}(M, \mathbb{Q}) \otimes \wedge^j H_{odd}(M, \mathbb{Q})$$

and

$$H_*(\wedge^n M, \mathbb{Q}) = \bigoplus_{i+j=n} \wedge^i H_{even}(M, \mathbb{Q}) \otimes Sym^j H_{odd}(M, \mathbb{Q}).$$

Therefore, for example, if $Sym^n M = 0$, then it follows that $Sym^n H_{even}(M, \mathbb{Q}) = 0$ and $\wedge^n H_{odd}(M, \mathbb{Q}) = 0$, which imply that $H_{even}(M, \mathbb{Q}) = 0$ and $\dim H_{odd}(M, \mathbb{Q}) < n$. Similar for the case when $\wedge^n M = 0$. □

Example 3.10. When M is isomorphic to the motive of a point, then it is evenly one dimensional. In fact, $\mathfrak{S}_2$ acts on $M \times M$ trivially, so $\wedge^2 M = 0$. When M is the motive of $\mathbb{P}^n$, then it is easy to see that M is a direct sum of $n+1$ motives each of which is isomorphic to the motive of a point, so M is evenly $n+1$-dimensional.

Let C be a smooth complete curve of genus g, and take a point $P \in C$. Define the motive $h^1(C)$ to be $(C, [\Delta_C] - [P \times C] - [C \times P])$. In the next section we will show that $Sym^{2g+1} h^1(C) = 0$, hence dim $h^1(C) = 2g$.

4. Motives of curves

Let C be a smooth complete curve of genus g, and $P \in C$ a point. Define the motive $h^1(C)$ to be $(C, [\Delta_C] - [P \times C] - [C \times P])$, where Δ_C is the diagonal of C. This section is devoted to prove Theorem 4.2, which says that $h^1(C)$ is oddly $2g$-dimensional.

This is actually a classical theorem by Shermenev [17], but I would like to include the proof of this theorem because it is a key point in this paper.

Let $\alpha = [\Delta_C] - [P \times C] - [C \times P] \in CH_*(C \times C)$, then

$$Sym^{(2g+1)} h^1(C) = (C^{(2g+1)}, \frac{1}{(2g+1)!} \sum_{\sigma \in \mathfrak{S}_{2g+1}} \overline{f_\sigma} \circ \alpha^{(2g+1)})$$

$$= (Sym^{(2g+1)} C, \frac{1}{(2g+1)!} (\pi \times \pi)_* \alpha^{(2g+1)})$$

where $\overline{f_\sigma}$ is as in Definition 3.2, $\alpha^{(2g+1)}$ as in Definition 3.3, and $\pi : C^{(2g+1)} \to Sym^{(2g+1)}$ the natural quotient morphism. So it is enough to prove that $(\pi \times \pi)_* \alpha^{(2g+1)}$ is 0.

Let us recall the relation between the symmetric product of a curve C and its Jacobian. Let $J(C)$ be the Jacobian variety of a curve C.

Proposition 4.1. *Let $P \in C$ be a point, and define $i : Sym^n C \to Sym^{n+1} C$ to be the morphism which sends $(x_1, x_2, \cdots, x_n)$ to $(x_1, x_2, \cdots, x_n, P)$. For $n > 2g-2$, $Sym^n C$ is a projective bundle over J so that $Sym^n C \cong \mathbb{P}(E_n)$, and one can choose the vector bundle E_n so that on $Sym^n C \cong \mathbb{P}(E_n)$, we have $c_1(\mathcal{O}(1)) \frown [Sym^{n+1} C] = i_*[Sym^n C]$.*

Proof: See [16]. $\square$

Theorem 4.2. *The motive $h^1(C)$ is oddly $2g$-dimensional.*

Proof: By Proposition 3.9, we have only to show that $Sym^{2g+1} h^1(C) = 0$.

Letting $p_i : Sym^{2g+1} C \times Sym^{2g+1} C \to Sym^{2g+1} C$ be the i-th projection ($i = 1, 2$), and $proj_{J \times J} : Sym^{2g+1} C \times Sym^{2g+1} C \to J \times J$ the natural projection, we need to show the following equality (*) in $CH_*(J \times J)$ for all $i \geq 0$ and $j \geq 0$.

$$(*)\ proj_{J \times J *}(c_1(p_1^* \mathcal{O}(1))^i \cdot c_1(p_2^* \mathcal{O}(1))^j \frown (\pi \times \pi)_* \alpha^{(2g+1)}) = 0.$$

Case 1) When $i = j = 0$, we have $\dim((\pi \times \pi)_* \alpha^{(2g+1)}) = 2g+1$, which is larger than the dimension of $J \times J$, so (*) holds in this case.

Case 2) When $i > 0$ or $j > 0$: In this case, we will prove that $c_1(p_1^* \mathcal{O}(1))^i \cdot c_1(p_2^* \mathcal{O}(1))^j \frown (\pi \times \pi)_* \alpha^{(2g+1)} = 0$. In fact, it is enough to show that $c_1(p_1^* \mathcal{O}(1)) \frown (\pi \times \pi)_* \alpha^{(2g+1)} = 0$. By the projection formula, it is same as $\pi_*(c_1(\pi^* p_1^* \mathcal{O}(1)) \frown p^{(2g+1)}) = 0$. By Proposition 4.1, $c_1(\pi^* p_1^* \mathcal{O}(1)) \frown [C^{2g+1} \times C^{2g+1}]$ is

$$\sum_{i=1}^{2g+1} [(C \times \cdots \times P \times C \times \cdots \times C) \times C^{2g+1}]$$

where P appears in the i-th place. So by symmetricity, we have only to show that the intersection product $[P \times C^{2g} \times C^{2g+1}] \cdot \alpha^{(2g+1)}$ is 0. By definition of $\alpha^{(2g+1)}$, the equality follows from the fact that the intersection product vanishes; $\alpha\cdot[P\times C]=0$. $\square$

Corollary 4.3. *Let C be a smooth curve, and $n>2g(C)$. If $c_1,\cdots,c_n\in CH_0(C)$ are 0-cycles with degree 0, then $Sym(c_1,\cdots,c_n)=0$.*

Proof: The 0-cycles c_i's are in $CH_0(h^1(C))$. The corollary follows from the fact $Sym(c_1,\cdots,c_n)\in CH_0(Sym^n(h^1(C)))$. $\square$

Corollary 4.4. *The motive of a smooth projective curve is finite dimensional.*

Proof: Theorem 4.2 says that $h^1(C)$ is oddly finite dimensional. On the other hand, it is easy to see that the motive $(C,[\Delta_C]-h^1(C))=(C,[P\times C])\oplus(C,[C\times P])$ is evenly two dimensional. (See Example 3.10 and Proposition 2.11.) $\square$

Remark 4.5. By a similar calculation as above, one can show that $Sym^{2g}h^1(C)=(Sym^{2g}C,\Gamma\times\Gamma)$ where $\Gamma\in CH_gSym^{2g}C$ is the unique cycle that satisfies $c_1(\mathcal{O}(1))\frown\Gamma=0$ and $proj_{J*}\Gamma=[J]$ where $proj_J:Sym^{2g}C\to J$ is the projection. It implies that top symmetric product of $h^1(C)$, namely $Sym^{2g}h^1(C)$, is isomorphic to the motive of a point by Proposition 2.11. For more properties of Γ, see [12].

5. Product of finite dimensional motives

In this section, we will prove that the product of finite dimensional motives are again finite dimensional. In particular, the motive of the product of curves is finite dimensional.

Let us start from recalling the construction of the Young symmetralizer. Let us follow the notation of [6]. Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ be a partition of N, consider the corresponding Young diagram, then Young tableau T is the Young diagram with numbers $\{1,2,\cdots,N\}$ in the boxes. Let $P\subset\mathfrak{S}_N$ be the subgroup consisting of the elements preserving the rows of T, and $Q\subset\mathfrak{S}_N$ the subgroup consisting of the elements preserving the columns of T. Define $a_{\lambda,T}:=\sum_{\sigma\in P}e_\sigma\in\mathbb{Q}\mathfrak{S}_N$ and $b_{\lambda,T}:=\sum_{\sigma\in Q}sgn(\sigma)e_\sigma\in\mathbb{Q}\mathfrak{S}_N$. Then the Young symmetralizer $c_{\lambda,T}$ is defined to be $c_{\lambda,T}=a_{\lambda,T}\cdot b_{\lambda,T}$. The Young symmetralizer $c_{\lambda,T}$ satisfies the following properties.

Proposition 5.1. *The Young symmetralizer $c_{\lambda,T}$ is an idempotent up to a constant, namely, we have $c_{\lambda,T}\cdot c_{\lambda,T}=n_\lambda\cdot c_{\lambda,T}$ for some non-zero integer n_λ. The image of the right multiplication of $c_{\lambda,T}$ in $\mathbb{Q}\mathfrak{S}_N$ is an irreducible representation of $\mathfrak{S}_N$, corresponding to the the partition λ.*

Proof: See Theorem 4.3 of [6]. $\square$

Proposition 5.2. *When M is a motive with $\wedge^{n+1}M=0$ and λ is a partition of some number N with $\lambda_1>n$, then for any Young tableau T subject to λ, we have $c_\lambda M^N=0$.*

Proof: It is enough to show that $b_{\lambda,T}M^N=0$. But it is easy to see that

$$b_{\lambda,T}M^N=\wedge^{\lambda_1}M\otimes\wedge^{\lambda_2}M\otimes\cdots\wedge^{\lambda_k}M.$$

Because we know that $\wedge^{\lambda_1}M=0$, we have $c_\lambda M^N=0$. $\square$

Now we need to study the relation between d_λ and $b_{\lambda,T}$'s.

Definition 5.3. *When* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$*, let* $Tab(\lambda)$ *be the set of* standard Young diagram*, namely, placing numbers so that increasing downward and rightward, subject to the partition* λ*.*

Theorem 5.4. *The number of standard Young diagrams subject to* λ*, namely* $\#Tab(\lambda)$*, is equal to the degree of the irreducible representation of* $\mathfrak{S}_N$ *corresponding to* λ*.*

Proof: See Theorem in page 169 of [13]. □

Definition 5.5. *Let* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ *and* $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ *be partitions of* N*. We say that* $\lambda < \mu$ *when there exists some* s $(0 \leq s)$ *such that* $\lambda_i = \mu_i$ *for* $i < s$ *and* $\lambda_s < \mu_s$*. Let* T *be a standard Young tableau subject to* λ*. Then for each* i $(1 \leq i \leq N)$*, the set of boxes occupied by the numbers* $\{1, 2, \cdots, i\}$ *forms a Young diagram. We denote the corresponding partition of* i *by* $T(i)$*. When* T *and* U *are standard Young tableaux subject to a partition* λ*, then we say that* $T < U$ *when for some* s $(0 \leq s)$*,* $T(i) = U(i)$ *for each* $i < s$ *and* $T(s) < U(s)$*.*

Lemma 5.6. *Let* T *and* U *be standard Young tableaux subject to* λ*. When* $T < U$*, there exists some* $i < j$ *such that* i *and* j *are in the same row in* T *and same column in* U*.*

Proof: By definition, $T(s) < U(s)$ for some s. Now one can find required $i < j \leq s$ by the standard argument. □

Proposition 5.7. *When* $T < U$ *are standard Young tableaux subject to* λ*, then we have* $c_{\lambda,T} \cdot c_{\lambda,U} = 0$*.*

Proof: It follows from Lemma 5.6 by the standard argument. See, for example, the proof of Lemma 4.23 of [6]. □

Proposition 5.8. *The projector* $d_\lambda \in \mathbb{Q}\mathfrak{S}_N$ *is a linear combination of monomials of* $c_{\lambda,T}$*'s.*

Proof: Let us start by ordering all the standard Young tableaux subject to λ by the order of Definition 5.5, say $T_1 < T_2 < \cdots < T_r$, where r is the number of the standard Young tableaux. We denote c_{λ,T_i} by c_i. We have $c_i \cdot c_j = 0$ for $i < j$ by Proposition 5.7. The c_i's are linealy independent by Proposition 5.1 and Proposition 5.7, hence by Theorem 5.4, they span the vector space $\{c \in \mathbb{Q}\mathfrak{S}_N | c \cdot \mathbb{Q}\mathfrak{S}_N$ is a direct sum of copies of representations corresponding to $\lambda\}$. As d_λ lies in this vector space, it is a linear combination of $c_{\lambda,T}$'s. □

Now combining Proposition 5.2 and Proposition 5.8, we have proved the following theorem for the evenly finite dimensional case.

Theorem 5.9. *Let* M *be a motive.When* $Sym^{n+1}M = 0$*,* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ *is a partition of* N*, and if* $\lambda_1 > n$*, then* $\mathbb{T}_\lambda M = 0$*. When* $\wedge^{n+1}M = 0$ *and* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ *is a partition of* N*, and if* $\lambda_{n+1} > 0$*, then* $\mathbb{T}_\lambda M = 0$*.*

Oddly finite dimensional case is similar (but one has to start from the construction of the Young symmetralizer as $c_{\lambda,T} := b_{\lambda,T} \cdot a_{\lambda,T}$) and left to the reader.

Proposition 5.10. *Let M and N be evenly or oddly finite dimensional motives. When M and N have the same parity (both even or both odd), then $M \times N$ is evenly finite dimensional. When M and N have different parity, then $M \times N$ is oddly finite dimensional. For both cases, we have* $\dim(M \times N) \leq \dim(M) \cdot \dim(N)$.

Proof: The proofs are similar for all cases, so we treat only the case when M and N are odd. Let $d = \dim(M)$ and $e = \dim(N)$. We have to show that $\wedge^{de+1}(M \times N) = 0$. The motive $\wedge^{de+1}(M \times N)$ is a submotive of $(M \times N)^{de+1} \simeq M^{de+1} \times N^{d+1} \simeq \oplus_{\lambda,\mu}(\mathbb{T}_\lambda M) \times (\mathbb{T}_\mu N)$. When W_λ and W_μ are irreducible representations corresponding to the partitions λ and μ, consider the decomposition $W_\lambda \otimes_{\mathbb{Q}\mathfrak{S}_{de+1}} W_\mu$ into irreducible representations, then it contains the alternating representation corresponding to the partition $(1, 1, \cdots, 1)$ if and only if λ and μ are dual partitions (corresponding to the flipping of the Young diagram). In terms of elements of the group ring, it means that considering $\mathbb{Q}(\mathfrak{S}_{de+1} \times \mathfrak{S}_{de+1}) = \mathbb{Q}\mathfrak{S}_{de+1} \otimes_{\mathbb{Q}} \mathbb{Q}\mathfrak{S}_{de+1}$ as a $\mathbb{Q}\mathfrak{S}_{de+1}$-module by diagonal imbedding $\mathfrak{S}_{de+1} \to \mathfrak{S}_{de+1} \times \mathfrak{S}_{de+1}$ sending σ to (σ, σ), we have the equation $d_{(1,1,\cdots,1)} \cdot (c_\lambda \otimes c_\mu) = 0$ if λ and μ are not dual partitions. By Proposition 5.8, we have $d_\lambda = \sum c'_i$ where each c'_i is a linear combination of monomials of $c_{\lambda,T}$'s, so we have $d_{(1,1,\cdots,1)} \cdot (d_\lambda \otimes d_\mu) = 0$ whenever λ and μ are not dual partitions. Let us write $M = (X, \alpha)$ and $N = (Y, \beta)$, and let $\hat{\lambda}$ be the dual partition of λ. We have

$$\begin{aligned}
\wedge^{de+1}(M \times N) &= ((X \times Y)^{de+1}, d_{(1,1,\cdots,1)} \circ (\alpha \times \beta)^{(de+1)}) \\
&= ((X \times Y)^{de+1}, d_{(1,1,\cdots,1)} \circ (\sum_\lambda d_\lambda \otimes \sum_\mu d_\mu) \circ (\alpha \times \beta)^{(de+1)}) \\
&= (x^{de+1} \times Y^{de+1}, d_{(1,1,\cdots,1)} \circ (\sum_\lambda (d_\lambda \otimes d_{\hat{\lambda}})) \circ (\alpha^{(de+1)} \times \beta^{(de+1)})) \\
&= \bigoplus_\lambda (X^{de+1} \times Y^{de+1}, d_{(1,1,\cdots,1)} \circ (d_\lambda \otimes d_{\hat{\lambda}}) \circ \alpha^{(de+1)} \times \beta^{(de+1)}).
\end{aligned}$$

So the vanishing of

$$(X^{de+1} \times Y^{de+1}, (d_\lambda \otimes d_{\hat{\lambda}}) \circ \alpha^{(de+1)} \times \beta^{(de+1)}) = \mathbb{T}_\lambda M \times \mathbb{T}_{\hat{\lambda}} N$$

for each λ implies the vanishing of $\wedge^{de+1}(M \times N)$. Because λ is a partition of $de + 1$, either $\lambda_1 > e$ (i.e., $\hat{\lambda}_{e+1} > 0$) or $\lambda_{d+1} > 0$, hence $\mathbb{T}_\lambda M = 0$ or $\mathbb{T}_{\hat{\lambda}} N = 0$ by Theorem 5.9, so we have the vanishing $\mathbb{T}_\lambda M \times \mathbb{T}_{\hat{\lambda}} N = 0$ for each λ. Therefore we have $\wedge^{de+1}(M \times N) = 0$, and $M \times N$ is evenly finite dimensional with $\dim(M \times N) \leq \dim M \cdot \dim N$. $\square$

Corollary 5.11. *When M and N are finite dimensional motives, then their product $M \times N$ is also finite dimensional.*

Now let us prove the finite dimensionality of the Albanese Kernel for the product of curves.

Lemma 5.12. *Assume that the base field is algebraically closed. Let C and D be smooth projective curves, and $\alpha \in CH_*(C \times D)$. Then α is in the Albanese Kernel if and only if $(h^1(C) \times h^1(D))_* \alpha = \alpha$.*

Proof: We have $Alb(C \times D) \simeq J(C) \times J(D)$. Let us identify $J(C)$ and $J(D)$ with the degree zero 0-cycles in $CH_0(C)$ and $CH_0(D)$ respectively. Define S :

$J(C) \oplus J(D) \oplus \mathbb{Z} \to CH_0(C \times D)$ by $S(c,d,n) := c \times [Q] + [P] \times d + n \cdot [(P,Q)]$ where P and Q are the fixed base points of C and D respectively, then S is a section of the $(alb, deg) : CH_0(C \times D) \to J(C) \oplus J(D) \oplus \mathbb{Z}$. For points $x \in C$ and $y \in D$, we have

$$S \circ (alb, deg)[(x,y)] = S\big(([x]-[P]),([y]-[Q]),1\big) = [(x,Q)] + [(P,y)] - [(P,Q)]$$

Now

$$h^1(C) \times h^1(D) = (C \times D, ([\Delta_C] - [P \times C] - [C \times P]) \times ([\Delta_D] - [Q \times D] - [D \times Q]))$$

So we have

$$(h^1(C) \times h^1(D))_*[(x,y)] = [(x,y)] - [(P,y)] - [(x,Q)] + [(P,Q)]$$

Therefore, we have $\alpha - (h^1(C) \times h^1(D))_* \alpha = S \circ (alb, deg)[(x,y)]$, which proves the lemma. $\square$

Lemma 5.13. *Let $c_1, \cdots, c_N$ be cycles in $CH_*(M)$. Then $c_1 \wedge \cdots \wedge c_N \in CH_*(\wedge^N M)$.*

Proof: Let us write $M = (X, \alpha)$, then $c_i = \alpha_* c_i$. We have $\wedge^N M = (X^N, 1/(N!) \cdot \sum sgn(\sigma)[\Gamma_\sigma] \circ \alpha^{(N)})$. So

$$CH_*(\wedge^N M) \ni 1/(N!)(\sum sgn(\sigma)[\Gamma_\sigma] \circ \alpha)_*(c_1 \times \cdots \times c_N)$$

$$= 1/(N!) \sum sgn(\sigma)[\Gamma_\sigma] \circ \alpha)_*(c_1 \times \cdots \times c_N)$$

$$= 1/(N!) \sum sgn(\sigma^{-1})(c_{\sigma(1)} \times \cdots \times c_{\sigma(N)})$$

$$= c_1 \wedge \cdots \wedge c_N$$

$\square$

Corollary 5.14. *Assume that the base field is algebraically closed. Let C and D be smooth projective curves, and $c_1, \cdots, c_N \in CH_0(C \times D)$ be 0-cycles in the Albanese Kernel. If $N > 4g(C) \cdot g(D)$, then $c_1 \wedge \cdots \wedge c_N = 0$.*

Proof: By Lemma 5.12, $c_i \in CH_0(h^1(C) \times h^1(D))$, so by Lemma 5.13, $c_1 \wedge \cdots \wedge c_N \in CH_0(\wedge^N(h^1(C) \times h^1(D)))$. But by Proposition 5.10, we have $\wedge^N(h^1(C) \times h^1(D)) = 0$ for $N > 4g(C) \cdot g(D)$. $\square$

Remark 5.15. One can actually prove that $c_1 \wedge \cdots \wedge c_N = 0$ for $N > g(C) \cdot g(D)$. For details, see [10]

6. Morphisms between finite dimensional motives

In this section, we analyze morphisms between finite dimensional motives. The key is the following proposition.

Proposition 6.1. *Let M and N be finite dimensional motives with different parity (namely, one is even and the other is odd), and $f : M \to N$ a morphism of motives. Then f is smash nilpotent, that is, when $k > \dim(M) \cdot \dim(N)$, the morphism $f^k : M^k \to N^k$ is zero.*

Proof: Decompose $M^k = \oplus_\lambda \mathbb{T}_\lambda(M)$ and $N^k = \oplus_\lambda \mathbb{T}_\lambda(N)$, and consider the induced morphism $\mathbb{T}_\lambda(M) \to M^k \xrightarrow{f^k} N^k \to \mathbb{T}_\mu N$. The morphism is given by $d_\mu \circ f^k \circ d_\lambda = d_\mu \circ d_\lambda \circ f^k$, so when $\lambda \neq \mu$, it is 0. Now we have only to show that each induced morphism $\mathbb{T}_\lambda(M) \to M^k \xrightarrow{f^k} N^k \to \mathbb{T}_\lambda N$ is 0. Let us say, M is odd and N is even, then because $k > \dim(M) \cdot \dim(N)$, either $\lambda_{\dim(M)+1} > 0$ or $\lambda_1 > \dim(N)$, so either $\mathbb{T}_\lambda(M) = 0$ or $\mathbb{T}_\lambda(N) = 0$ by Theorem 5.9, hence $f^k = 0$. □

Corollary 6.2. *If a motive M is finite dimensional both evenly and oddly, then $M = 0$.*

Proof: The identity morphism $id_M : M \to M$ can be considered as a morphism between motives of different parity, so it is smash nilpotent. By Proposition 2.16, it is nilpotent, so we have $id_M = 0$, hence $M = 0$. □

The decomposition of a finite dimensional motive into even part and odd part is by no means unique. For example, in our definition of $h^1(C)$ (namely the odd part of the motive of C), it does depend on the choice of the base point P. But we can show that the isomorphism classes of the even part and the odd part are well-defined as follows.

Proposition 6.3. *Let M be a finite dimensional motive, and $M = M_+ \oplus M_-$ and $M = N_+ \oplus N_-$ be two decompositions into the even part and odd part; M_+ and N_+ are evenly finite dimensional, and M_- and N_- are oddly finite dimensional. Then we have $M_+ \simeq N_+$ and $M_- \simeq N_-$.*

Proof: We will prove that $M_+ \simeq N_+$. Let $e : M \to M$ be the projector to M_+, and $f : M \to M$ the projector to N_+. Then $1 - f : M \to M$ is the projector to N_-, so by Proposition 6.1, the morphism $(1 - f)e$ is smash nilpotent. By Proposition 2.16, it is nilpotent, so we have $\big((1 - f)e\big)^n = (e - fe)^n = 0$ for some $n > 0$. Expanding it out, we get

$$e = (\text{some expression of } e \text{ and } f)fe$$

using the fact that e is idempotent (be careful, e and f does not necessarily commute). We can consider fe to be the morphism from M_+ to N_+, and the formula above implies that (some expression of e and f) is the section from N_+ to M_+, which makes M_+ an direct summand of N_+, say $M_+ \oplus K \simeq N_+$. This implies that $\dim(N_+) = \dim(M_+) + \dim(K)$. But similarly we can prove that $\dim(M_+) \geq \dim(N_+)$, which implies that $\dim(K) = 0$, hence $K = 0$. Therefore we have obtained the isomorphism $M_+ \simeq N_+$. Similarly for $M_- \simeq N_-$. □

Definition 6.4. *Let $M = M_+ \oplus M_-$ be a finite dimensional motive with M_+ evenly finite dimensional, M_- oddly finite dimensional. Then we define* the dimension of M *to be* $\dim(M_+) + \dim(M_-)$. *It is well defined by Proposition 6.3.*

Definition 6.5. *Let $f : M \to N$ be a morphism of motives. The morphism f is called* surjective *if, for any smooth complete variety Y, the induced morphism $(f \times id_Y)_* : CH_*(M \times Y) \to CH_*(N \times Y)$ is surjective.*

Remark 6.6. When M is the motive of a variety X, and N is the motive of a variety Y, and f is induced by a proper surjective morphism $F : X \to Y$, then f is surjective (see Proposition 1.3 of [7]).

Lemma 6.7. *Let $f : X \vdash Y$ be a correspondence. Consider f as an element of $CH_*(X \times Y)$, and let $f^t \in CH_*(Y \times X)$ be its dual correspondence, or the transpose. For any given smooth complete variety Z, let $f_* : CH_*(X \times Z) \to CH_*(Y \times Z)$ be the morphism induced by the correspondence f. Considering $c \in CH_*(X \times Z)$ as a correspondence from X to Z, we have $f_*(c) = c \circ f^t$.*

Proof: Straightforward. $\square$

Lemma 6.8. *Let $M = (X, \alpha)$ and $N = (Y, \beta)$. The morphism $f : M \to N$ is surjective if and only if there exists a cycle $c \in CH_*(M \times Y)$ such that $f_*(c) = \beta$ in $CH_*(Y \times Y)$. If f is surjective, c^t determines a section of f.*

Proof: We consider β as a cycle in $CH_*(Y \times Y)$. By Definition 6.5, when f is surjective, we have c with $f_*(c) = \beta$. Conversely assume that we have such $c \in CH_*(M \times Y) \subset CH_*(X \times Y)$. For any variety Z, and $\gamma \in CH_*(X \times Z)$, considering c as a correspondence from X to Y, we have

$$f_*(\gamma \circ c) = \gamma \circ c \circ f^t = \gamma \circ (f_*(c)) = \gamma \circ \beta = \gamma$$

By Lemma 6.7, the right inverse map $\_ \circ c$ is $(c^t)_*$. $\square$

Proposition 6.9. *Let $f : M \to N$ be a surjective morphism of motives, and assume that M is finite dimensional. Then N is also finite dimensional.*

Proof: When M is oddly or evenly finite dimensional, say evenly finite dimensional, then $\wedge^n M \to \wedge^n N$ is also surjective, hence N is also finite dimensional. So we need only to show that N also decomposes into $N_+ \oplus N_-$ so that $M_+ \to N_+$ and $M_- \to N_-$ are surjective. Let us write $M = (X, \alpha)$ and $N = (Y, \beta)$, and consider f as an correspondence from X to Y with $f \circ \alpha = f = \beta \circ f$. By Lemma 6.8, there exists a cycle $c \in CH_*(M \times Y)$ such that $f_*(c) = \beta$. Now decompose M into $M_+ \oplus M_-$, and let e_+ and e_- be the corresponding idempotents, then we have $\alpha = e_+ + e_-$, hence defining $c_+ := c \circ e_+$ and $c_- := c \circ e_-$, we have $c_+ + c_- = c$. Define $r_+ := f_*(c_+)$ and $r_- := f_*(c_-)$, then $r_+ + r_- = \beta$. By Lemma 6.7, we have $r_+ = c_+ \circ f^t = c \circ e_+ \circ f^t$, and $r_- = c_- \circ f^t = c \circ e_- \circ f^t$. Now considering r_+ and r_- as morphisms from N to N, we claim that $r_+ \circ r_- = r_+ - r_+^2$ is nilpotent. In fact, it is $c \circ e_+ \circ f^t \circ c \circ e_- \circ f^t$, which factors through M_+ and M_-. By Proposition 6.1, $r_+ \circ r_-$ is a smash nilpotent, hence nilpotent. Let us say, $(r_+ - r_+^2)^k = 0$. Define β_+ to be $\left(1_N - (1_N - r_+)^k\right)^k$. We will show that β_+ is an idempotent and gives a required decomposition of even part and odd part of N. Let us take

$$G(r_+) := \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} r_+^{i-1}$$

then $\beta_+ = \left(r_+ \cdot G(r_+)\right)^k$, and we have

$$\beta_+^2 = \left(1_N - (1_N - r_+)^k\right)^k \cdot r_+^k \cdot G(r_+)^k$$

and using the fact $(1_N - r_+)^k \cdot r_+^k = 0$, we have $(1_N - (1_N - r_+)^k) \cdot r_+^k = r_+^k$, hence $(1_N - (1_N - r_+)^k)^k \cdot r_+^k = r_+^k$, therefore $\beta_+^2 = \beta_+$.

Now we have to show that M_+ surjects onto (Y, β_+) and M_- surjects onto (Y, β_-). By construction, β_+ is a linear combination of monomials of r_+ with no constant term, and because $r_+ = c \circ e_+ \circ f^t$, $r_+^k = f_*\{(c \circ e_+ \circ f^t)^{k-1} \circ c \circ e_+\}$ by Lemma 6.7, hence by Lemma 6.8, the morphism $M_+ \to (Y, \beta_+)$ is surjective. For $\beta_- := \beta - \beta_+$, we calculate $\beta_- = 1_N - (1_N - r_-^k)^k$, which is a linear combination of $r_- = c \circ e_- \circ f^t$ without contant term, so similarly to the β_+-case, we can show that $M_- \to (Y, \beta_-)$ is surjective as required. $\square$

7. Finite Dimensionality Conjecture and its Consequences

In this section, we study the consequences of the following conjecture.

Conjecture 7.1. *(Finite dimensionality) Let X be a smooth variety. Then the motive of X is finite dimensional.*

Proposition 7.2. *Let $f : M \to M$ be a morphism of motives, and assume that M is either evenly or oddly finite dimensional. Then*

i) The morphism f satisfies $G(f) = 0$, where $G(t) \in \mathbb{Q}[t]$ is some monic polynomial.

ii) If f is homologically trivial, namely $H_*(f) = 0$, then the morphism f is nilpotent.

Proof: Let us take $f^{\oplus n} : M^{\oplus n} \to M^{\oplus n}$ with $n > \dim(M)$. When M is evenly finite dimensional, consider $d_{(1,\cdots,1)} \circ f^{(n)} : M^n \to M^n$. Then it induces a morphism to $\wedge^n M$, hence it is 0. Now take projectors $\pi_i : M^n \times M^n \to M \times M$ sending $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ to (x_i, y_i), considering $d_{(1,\cdots,1)} \circ f^{(n)}$ as an element of $CH_*(M^n \times M^n)$, and we get

$$\pi_{1*}(d_{(1,\cdots,1)} \circ f^{(n)} \cdot \pi_2^*([\Delta_M]) \cdots \pi_n^*([\Delta_M])) = 0.$$

For each $\sigma \in \mathfrak{S}_n$, define $orb(\sigma)$ to be the set of orbits of σ in $\{1, 2, \cdots, n\}$, and for each $s \in orb(\sigma)$, define $\ell(s)$ to be the number of elements in s. Define $s_\sigma(1)$ to be the orbit of σ that contains 1. Now the left hand side of the equation above is expanded out as follows:

$$\sum_\sigma f^{\ell(s_\sigma(1))} \cdot \prod_{s \in orb(\sigma) \setminus \{s_\sigma(1)\}} deg(f^{\ell(s)} \cdot [\Delta_M]) \cdot sgn(\sigma) = 0$$

Similarly, when M is oddly finite dimensional, we have

$$\sum_\sigma f^{\ell(s_\sigma(1))} \cdot \prod_{s \in orb(\sigma) \setminus \{s_\sigma(1)\}} deg(f^{\ell(s)} \cdot [\Delta_M]) = 0$$

So f is a root of some polynomial. To see that this polynomial is non-trivial, we have to see that its top degree term has non- zero coefficients. Let us compute the coefficient of f^n, the highest degree term. It comes from σ's with only one orbit so that $\ell(s_\sigma(1)) = n$, and the contribution from each σ is $(-1)^{n+1}$ for the even case and 1 for the odd case, so the coefficient of f^n is $(n-1)! \cdot (-1)^{n+1}$ for the even case and $(n-1)!$ for the odd case, so the polynomial is non-trivial and we have proved the part i). For the part ii), when f is homologically equivalent to 0, then we have $deg(f^{\ell(s)} \cdot [\Delta_M]) = 0$, so the only non-zero terms are the top degree terms, which means $f^n = 0$ in $\mathbb{Q}$-coefficients. $\square$

Corollary 7.3. *When M is a finite dimensional motive and $H^*(M) = 0$, then $M = 0$.*

Proof: We have a decomposition $M = M_+ \oplus M_-$, then by assumption, id_{M_+} and id_{M_-} are homologically equivalent to zero, so they are nilpotent by Proposition 7.2, which implies $M_+ = 0$, and $M_- = 0$, hence $M = 0$. □

Corollary 7.4. *Let M be a finite dimensional motive, then* $\dim(M) = \dim_{\mathbb{Q}} H^*(M)$.

Proof: We may assume that either M is even or odd. By Proposition3.9, we have only to show that $\dim(M) \leq \dim_{\mathbb{Q}} H^*(M)$. Let us say M is even, and $\dim(M) = n$, then we have $\wedge^n M \neq 0$, and $H^*(M)$ is concentrated in the even dimensional part by Proposition 3.9. By Corollary 7.3, $H^*(\wedge^n(M)) = \wedge^n_{\mathbb{Q}}(H^*(M)) \neq 0$, which implies that $\dim_{\mathbb{Q}} H^*(M) \geq \dim(M)$. Similar for the odd case. □

Proposition 7.5. *Let M be a finite dimensional motive, and $f : M \to M$ a morphism such that $H^*(f) = 0$. Then f is nilpotent.*

Proof: Decompose $M = M_+ \oplus M_-$, and let e_+ and e_- be the projectors to M_+ and M_- respectively. As $id_M = e_+ + e_-$, we have a decomposition of f into $f = 1f1 = (e_+ + e_-)f(e_+ + e_-) = e_+fe_+ + (e_+fe_- + e_-fe_+) + e_-fe_-$. Let us write $f_+ := e_+fe_+$, $f_\pm := e_+fe_- + e_-fe_+$, and $f_- := e_-fe_-$. As $e_+e_- = e_-e_+ = 0$, we have $f_+f_- = f_-f_+ = 0$. As e_+fe_- and e_-fe_+ are smash nilpotents by Proposition 6.1, their sum $f_\pm$ is also smash nilpotent by Proposition 2.15.

Now consider $f^n = f \circ \cdots \circ f = (f_+ + f_\pm + f_-) \circ \cdots \circ (f_+ + f_\pm + f_-)$ and expand it out. This is a sum of monomials of f_+, $f_\pm$ and f_- of degree n in some order. We will show that when n is very large, all these monomials are 0.

As $f_\pm$ is smash nilpotent, so there exists some number N such that $f_\pm \times \cdots \times f_\pm = 0$ (N times). So in the composition of f_+, $f_\pm$ and f_-, if $f_\pm$ appears more than $N-1$ times, then the composition is 0 by Proposition 2.16. Hence we may assume that in the composition, $f_\pm$ appears at most $N-1$ times. Between two $f_\pm$'s (or before the first $f_\pm$ or after the last $f_\pm$), if both f_+ and f_- show up, then the composition is 0, because $f_+ \circ f_- = f_- \circ f_+ = 0$, so they are some power of f_+ or f_-. So for example, typical candidate for non-zero composition is as follows, with $r \leq N$ and $r - 1 + \sum_{i=1}^r k_i = n$:

$$f_+^{k_1} \circ f_\pm \circ f_-^{k_2} \circ f_\pm \circ f_-^{k_3} \circ \cdots \circ f_\pm \circ f_+^{k_r}$$

Because f_+ and f_- are nilpotents by Proposition 7.2, there is some number L such that $f_+^L = 0$ and $f_-^L = 0$. If $n \geq NL$, then in the expression as above, one of k_i must be larger than or equal to L, which implies that $f_-^{k_i}$ or $f_+^{k_i}$ is 0, hence all such compositions vanish. Therefore, $f^{NL} = 0$ and $f : M \to M$ is a nilpotent. □

Proposition 7.6. *Let X be a smooth variety and assume that the cycle map cl : $CH_*(X) \to H_*(X, \mathbb{Q})$ is surjective. Then the cycle map is bijective.*

Proof: Let $\{\alpha_1, \cdots, \alpha_k\}$ be a basis of the $\mathbb{Q}$-vector space $H_*(X, \mathbb{Q})$, and $\{\hat{\alpha}_i, \cdots, \hat{\alpha}_k\}$ the dual basis by the cup product, identifying $H_*(X, \mathbb{Q})$ with $H^*(X, \mathbb{Q})$ by Poincaré duality. Then we have

$$\sum_{i=1}^k \alpha_i \times \hat{\alpha}_i = [\Delta_X] \in H_*(X \times X, \mathbb{Q}).$$

Pick β_i and $\hat{\beta}_i$ in $CH_*(X)$ so that $cl(\beta_i) = \alpha_i$ and $cl(\hat{\beta}_i) = \hat{\alpha}_i$, and consider the correspondence $f = \sum_{i=1}^k \beta_i \times \hat{\beta}_i : X \vdash X$. By Proposition 7.5, the correspondence $[\Delta_X] - f$ is nilpotent. On the other hand, direct calculation shows that f is an idempotent, hence $0 = ([\Delta_X] - f)^n = [\Delta_X] - f = 0$ and $[\Delta_X] = f = \sum_{i=1}^k \beta_i \times \hat{\beta}_i$. For any $b \in CH_*X$, we have $b = [\Delta_X]_* b = \sum (b \cdot \beta_i) \cdot \hat{\beta}_i$, so $\hat{\beta}_i$'s span the $\mathbb{Q}$-vector space CH_*X. As $\hat{\beta}_i$'s are linearly independent, they form a basis of CH_*X, hence the cycle map is bijective. □

Corollary 7.7. *Let X be a surface with $p_g = 0$. If the motive of X is finite dimensional, then the Albanese Kernel is 0 in $CH_0(X)_{\mathbb{Z}}$.*

Proof: We may assume that $q = 0$ by [3]. It is easy to see that for such surfaces, the cycle map is surjective, so by Proposition 7.6, $CH_0(X) = \mathbb{Q}$, hence the Albanese Kernel is torsion. But by Roitman's theorem [15], non-zero torsion cycles cannot be in the Albanese kernel, so the Albanese kernel is 0. □

Corollary 7.8. *Let M be a finite dimensional motive, and $f : M \to M$ a homological idempotent, namely $f - f^2$ is homologically equivalent to zero. Then there exists an idempotent $F : M \to M$ such that F is homologically equivalent to f.*

Proof: By Proposition 7.5, the morphism $f - f^2$ is a nilpotent, say, $(f - f^2)^N = 0$. Define the morphism $F : M \to M$ by $F = \left(1 - (1-f)^N\right)^N$. Then direct calculation shows that $F - f$ is homologically equivalent to zero, and $F^2 = F$. □

Corollary 7.9. *Let M be a finite dimensional motive, and $f : M \to M$ be a morphism. Then there exists a non-zero polynomial $G(t) \in \mathbb{Q}[t]$ such that $G(f) = 0$. The set of the eigenvalues of f is same as the set of the eigenvalues of $H^*(f)$.*

Proof: Consider the characteristic polynomial of $H^*(f) : H^*(M) \to H^*(M)$, say $\varphi(t) \in \mathbb{Q}[t]$. Then $\varphi(f) : H^*(M) \to H^*(M)$ is 0, so some of its power is zero by Proposition 7.5. □

Theorem 7.10. *Let X be an n-dimensional variety that satisfies the following conditions. 0) The motive of X is finite dimensional. 1) The Hodge component $H^{i,j}(X, \mathbb{C}) = 0$ for $|i - j| > 1$. 2) The cycle map $cl_i : CH_i X \to H^{i,j}(X, \mathbb{C}) \cap H^{2i}(X, \mathbb{Q})$ is surjective. 3)For each i, there exist an Abelian variety J_i and a correspondence $\Gamma_i : X \to J_i$ which induces an isomorphism $\Gamma_{i*} : H^{2n-2i-1}(X, \mathbb{Q}) \simeq H^1(J_i, \mathbb{Q})$.*

Then the Chow group of X is representable.

Remark 7.11. In the condition 3), J_i is isogenous to the intermediate Jacobian, and the morphism $alb \circ \Gamma_{i*} : CH_i(X) \to CH_0 J_i \to J_i$ is isogenous to the generalized Abel-Jacobi map.

Remark 7.12. The condition 0) is our finite dimensional conjecture, the condition 2) is Hodge conjecture, and 3) follow from the generalized Hodge conjecture when we have 1), so conjecturally the representability of Chow group follows from 1).

Lemma 7.13. *Under the assumption 3) of Theorem 7.10, there exists a correspondence $\varphi_i : J_i \vdash X$ such that $\Gamma_i \circ \varphi_i : J_i \to J_i$ is an idempotent and the motive determined by the idempotent is $h_1(J_i)$, namely $H_*(h_1(J_i)) = H_1(J_i)$.*

Proof: One can construct $h_1(J_i)$ by eigenvalue decomposition by Murre-Deninger [4], say $h_1(J_i) = (J_i, \alpha)$, then $\alpha \circ \Gamma_i : X \vdash h_1(J_i)$ is surjective. By [11], the motive $h_1(J_i)$ is projective in the category of bivariant sheaves (the motive of a smooth varieity is projective, and $h_1(J_i)$ is its direct summand), which implies that there is a section $\varphi : h_1(J_i) \to X$. We have only to define φ_i to be $\varphi \circ \alpha$. □

Proof: (of Theorem 7.10) Let us take the correspondence $p_i = \varphi_i \circ \Gamma_i : X \vdash X$, then easy calculation shows that $p_i \circ p_i$ is an idempotent, and its cohomology group is $H^{2n-2i-1}(X, \mathbb{Q}) \simeq H^1(J_i, \mathbb{Q})$. Define the correspondence $e_{odd} : X \vdash X$ to be $\sum_{i=0}^{n-1} p_i \circ p_i$. Now, let c_j be a $\mathbb{Q}$-basis of $H_{even}(X, \mathbb{Q})$, and $\hat{c}_j$ its dual basis by intersection products. Pick $d_j, \hat{d}_j \in CH_*(X)$ so that $cl(d_j) = c_j$ and $cl(\hat{d}_j) = \hat{c}_j$ respectively. We define e_{even} to be $\sum_j d_j \times \hat{d}_j \in CH_n(X \times X)$, and take $e = e_{even} + e_{odd}$. Then the morphism $e_* : H_*(X, \mathbb{Q}) \to H_*(X, \mathbb{Q})$ is identity, hence $[\Delta_X] - e$ is nilpotent, and the correspondence e is invertible and $CH_*(X) = e_* CH_* X$. Now from the construction, e_{even} is idempotent and $CH_*((X, e_{even}))$ is a finite dimensional $\mathbb{Q}$-vector space. Also by the construction, p_i factors through $h_1(J_i)$, and it identifies $h_1(J_i)$ as a submotive of X. The homology action of the correspondence $p_i \circ p_j$ for $i \neq j$ is 0, hence $p_{i*}(CH_*(h_1(J_j))) = 0$, so we have $e_{odd*} CH_*(X) = \oplus_i CH_*(h_1(J_i)) = \oplus_i J_i$. Therefore, $CH_*(X) = \oplus_j \mathbb{Q}\hat{c}_j \oplus_i J_i$ is representable. □

8. Non-vanishing of the wedge products of the Jacobian

In this section, we will prove the non-vanishing of the wedge products of 0-cycles with degree 0 on a curve.

Proposition 8.1. *Let C be a smooth curve. Assume that there exists a number $r > 0$ such that for 0-cycles $a_1, a_2, \cdots, a_r \in CH_0 C$ with $\deg a_i = 0$, we have $a_1 \wedge \cdots \wedge a_r = 0$. Then for all $N > 2g(C)(r-1)$ and 0-cycles $b_1, \cdots, b_N \in CH_0(C)$ with $\deg b_i = 0$, we have $b_1 \times \cdots \times b_N = 0$.*

Proof: Recall that $Sym(c_1, \cdots, c_{2g+1}) = 0$ when c_i's are elements of $CH_0(C)$ with $\deg c_i = 0$. The rest of the proof goes exactly similarly to the proof of Proposition 6.1, and left to the reader. □

Let J be an Abelian variety. Define the morphism $\varphi_{(n,i)} : J \times \cdots \times J \to J \times \cdots \times J$ by sending $(P_1, \cdots, P_k)$ to $(P_1, \cdots, P_{i-1}, n \cdot P_i, P_{i+1}, \cdots, P_k)$ where $n \cdot P_i = \overbrace{P_i + P_i + \cdots + P_i}^{n \text{ times}}$. Recall that we have an eigenspace decomposition of the Chow group $CH_0(J \times \cdots \times J)$, say $a \in CH_{0,(d_1,\cdots,d_k)}(J \times \cdots \times J)$ when we have $\varphi_{(n,i)*} a = n^{d_i} \cdot a$ for each n and i. By theorems of Beauville [1] and Murre-Deninger [4], we have $CH_0(J \times \cdots \times J) = \oplus CH_{0,(d_1,\cdots,d_k)}(J \times \cdots \times J)$ where $(d_1, \cdots, d_k)$ runs over all the integers $0 \le d_1, \cdots, d_k \le g(C)$.

Definition 8.2. *Define $CH_{0,(d_1,\cdots,d_k)}(J \times \cdots \times J)$ as above. Also define $CH_{0,d}(J)$ to be $CH_{0,(d)}(J)$, omitting k when $k = 1$.*

Proposition 8.3. *The Pontrjagin product morphism*

$$CH_{0,(1,1,\cdots,1)}(J \times \cdots \times J)^{\otimes g} \to CH_{0,(g,\cdots,g)}(J \times \cdots \times J)$$

is surjective.

Proof: Consider the exterior product map $CH_0(J)^{\otimes k} \to CH_0(J \times \cdots \times J)$. Because the image contains the cycles of the form $[P_1] \times \cdots \times [P_k]$, which generates $CH_0(J \times \cdots \times J)$, this map is surjective. Decomposing this map into eigenspaces, we obtain the surjection $CH_{0,g}(J)^{\otimes k} \to CH_{0,(g,\cdots,g)}(J \times \cdots \times J)$. Now let I be the degree 0 cycles in $CH_0(J)$, then by [1], $I = \oplus_{d>0} CH_{0,d}(J)$, and by [2], the Pontrjagin product gives the bijection $I^{*g} = CH_{0,g}(J)$, so combining, again decomposing into eigenspaces, we have the surjection $CH_{0,1}(J)^{\otimes kg} \to CH_{0,(g,\cdots,g)}(J \times \cdots \times J)$. This surjection factors through $CH_{0,(1,1,\cdots,1)}(J \times \cdots \times J)^{\otimes g}$, hence we obtain the desired surjection. □

Proposition 8.4. *Let C be a smooth curve, and J its Jacobian variety. Then for any $P \in J$, there exist morphisms $\phi_1, \cdots, \phi_r : C \to J$ and 0-cycles $a_1, \cdots, a_r \in CH_0C$ with $\deg a_i = 0$ such that*

$$[P] - [0] = \sum_{i=1}^{r} \phi_{i*} a_i,$$

where $0 \in J$ is the identity.

Proof: Let us take a point $Q \in C$ and fix it. Consider the Abel-Jacobi map $\phi : C \to J$ which sends Q to 0, then it induces the canonical morphism $S^g C \to J$. Because the morphism $S^g C \to J$ is surjective, we have points $b_1, \cdots b_r \in C$ such that $P = \sum_{i=1}^{r} \phi(b_i)$. We have only to define $\phi_i(x) = \phi(x) + \phi(b_1) + \phi(b_2) + \cdots + \phi(b_{i-1})$ and $a_i = [b_i] - [Q]$. □

Theorem 8.5. *Let C be a non-rational smooth curve, and $r > 0$ an integer. Then for some $a_1, \cdots, a_r \in CH_0C$ with $\deg a_i = 0$, we have $a_1 \wedge \cdots \wedge a_r \neq 0$ in $CH_0(C \times \cdots \times C$.*

Proof: Assume that there is some integer $r > 0$ such that $a_1 \wedge \cdots \wedge a_r = 0$ for all $a_1, \cdots, a_r \in CH_0(C)$ with $\deg a_i = 0$, and we will get a contradiction. By Proposition 8.1, it follows that $b_1 \times \cdots \times b_N = 0$ for all $b_1, \cdots, b_N \in CH_0C$ with degree 0, $N > 2g(r-1)$. Let J be the Jacobian variety of C, and $I \subset CH_0(J)$ the subgroup of degree 0 cycles. Because $CH_0(J)$ is generated by the cycles $[P]$ $(P \in J)$ and because we have an exact sequence $0 \to I \to CH_0(J) \to \mathbb{Z} \to 0$, I is generated by the cycles $[P] - [0]$ where $0 \in J$ is the identity. Because $b_1 \times \cdots \times b_N = 0$ for $N > 2g(r-1)$, Proposition 8.4 implies that $CH_{0,(1,\cdots,1)}(J \times \cdots \times J) = 0$ where J repeats N times. By Proposition 8.3, it implies that $CH_{0,(g,\cdots,g)}(J \times \cdots \times J) = 0$, and letting $A = J \times \cdots \times J$ be the product Abelian variety, we have $CH_{0,Ng}(A) = 0$. But according to[2], we should have $CH_{0,Ng}(A) \neq 0$, contradiction. □

9. Examples

In this section, we take a look at several examples.

Example 9.1. Let A be an abelian variety, and $\iota : A \to A$ the inverse map of the group variety structure. Define the group G to be $\{id_A, \iota\}$, and take $X = A/G$. The variety X has only quotient singularities, and as far as we are concerned with the Chow groups with rational coefficients, we can treat X as if it is smooth by [19]. Any abelian variety is a quotient of a Jacobian variety, which in turn is a surjective image of a product of curves, so the motives of X and A are finite dimensional. The action of ι on $H^i(A, \mathbb{Q})$ is the multiplication by $(-1)^i$, so the motive of X is evenly finite dimensional, in fact its dimension is $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) = 2^{2n-1}$ where n is the dimension of X. When $c_1, \cdots, c_N \in CH_*(X)$ are algebraic cycles with $N > 2^{2n-1}$ (no restrictions for the cycles!), then we have $c_1 \wedge \cdots \wedge c_N = 0$.

Example 9.2. Let ζ be a primitive 10-th root of unity. Let $C \to \mathbb{P}^1$ be the double covering branching at $0, 1, \zeta^2, \zeta^4, \zeta^8$, and ∞. Then C is a smooth curve of genus 2. The curve C is the normalization of the closure of $y^2 = x^6 - x$ in $\mathbb{C}^2 \subset \mathbb{P}^2$. The $\mathbb{C}$-vetor space $H^0(C, \Omega_C)$ is spanned by dx/y and xdx/y. Let $G = \mathbb{Z}/10\mathbb{Z}$ be the abelian group of order 10 with a generator g, and let G act on C by $g(x, y) = (\zeta^2 x, \zeta y)$. Let us consider the product $C \times C \times C$, let G act on the product diagonally, and define X to be the quotient $C \times C \times C/G$. The variety X has only quotient singularities, hence we can treat it as if it is smooth [19]. Now calculating the eigenvalues of g in $H^{1,0}(C \times C \times C)$, $H^{2,0}(C \times C \times C)$, $H^{3,0}(C \times C \times C)$ and $H^{2,1}(C \times C \times C)$, one can check that no eigenvalue is 1, hence $H^{i,j}(X) = 0$ whenever $i \neq j$. Hodge conjecture holds for threefolds, hence the cycle map is surjective, therefore by Proposition 7.6, the Chow group of X is isomorphic to the homology group, $CH_*(X) \simeq H_*(X, \mathbb{Q})$. We have $H^0(X, 2K) \neq 0$, so at least the Kodaira dimension of X is non-negative.

Example 9.3. Take the elliptic curve C defined by $y^2 z = x^3 - z^3$ in $\mathbb{P}^2$. Let $G = \mathbb{Z}/3\mathbb{Z} = \{1, g, g^2\}$ be a group and define the action of G on C by $g(x, y, z) = (\omega x, y, z)$ where ω is a primitive cubic root of unity. The space of 1-forms $H^0(C, K)$ is spanned by dx/y, and g acts on it by multiplication by ω. Let G act on $C \times C \times C$ diagonally, and define X to be the quotient $C \times C \times C/G$. Then by calculating the eigenvalues of g on $H^*(C \times C \times C)$, one obtains $H^{1,0}(X) = H^{2,0}(X) = H^{2,1}(X) = 0$, but $H^{3,0} \neq 0$. In this case, letting $M = M_+ \oplus M_-$ be the motive of X decomposed into the even dimensional part and the odd dimensional part, M_+ is isomorphic to finitely many points, which situation is similar to the motive of a curve, but we cannot hope to represent $CH_*(X)$ by a scheme. One might be able to study its Chow group by analyzing the rational curves on the symmetric product $Sym^n X$, which seems to be a very interesting question.

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