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Course and Project Description

The Mahler measure of a non-zero Laurent polynomial $P \in \mathbf{C}[x_1, x_1^{-1}, \cdots, x_N, x_N^{-1}]$ is defined by

$$m(P) = \int_0^1 \cdots \int_0^1 \log \left| P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| d\theta_1 \cdots d\theta_n.$$

This quantity arises naturally, for example, in the computation of heights of subvarieties of tori.

In these lectures I will describe how the Mahler measure is related to the regulator of the Bloch–Beilinson conjectures. Concretely, we will discuss what is the theoretical framework behind identities like the one discovered by Smyth

$$m(x + y + 1) = L'(\chi, -1),$$

where χ is the quadratic Dircihlet character of conductor 3. Our tour will involve K-theory, the dilogarithm and hyperbolic 3-manifolds.

Problems:

1. Let $E = X_1(11)$, an elliptic curve defined over $\mathbf{Q}$ of conductor 11 with minimal Weierstrass model $y^2 + y = x^3 - x^2$. Find two rational functions f and g on E such that the symbol $\{f, g\}$ is a non-torsion integral element of $K_2(E)$ (See the article by D. Ramakrishnan, "Regulators, Algebraic Cycles, and Values of L-functions", in the book Algebraic K-theory and Algebraic Number Theory, Contemporary Mathematics 83, pp. 183–310, (1989), Amer. Math. Soc. Providence, R.I. (M. R. Stein and R. K. Dennis eds.).)
2. Let

$$P = (x + y + 1)(x + 1)(y + 1) + xy$$
$$Q = y^2 + (x^2 + 2x - 1)y + x^3.$$

Prove that

$$m(P) = 7/5m(Q)$$