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Arizona Winter School 2002

Project description

Compute the $\zeta^{(p)}(s,t)$ mentioned above, or make an educated guess, and verify that they satisfy the identities known for the $\zeta(s,t)$.

The "known identities" I have in mind are:

(a) Let Δ be the coproduct on $\mathbb{Q} \ll e_0, e_1 \gg$ for which the e_i are primitive: $\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$. It induces a coproduct on $\mathbb{Q} \ll e_0, e_q \gg / \mathbb{Q} \ll e_0, e_1 \gg e_0 + e_1 \mathbb{Q} \ll e_0, e_1 \gg$. In this quotient,

$$g_{\mathbb{R}} := \sum \zeta(s_1, \ldots, s_r) e_0^{s_1-1} e_1 \ldots e_0^{s_r-1} e_1$$

is group like: $\Delta g_{\mathbb{R}} = g_{\mathbb{R}} \otimes g_{\mathbb{R}}$.

(b) $\zeta(s)\zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t)$ (an instance of $\sum\limits_{n,m} = \sum\limits_{n>m} + \sum\limits_{m>n} + \sum\limits_{n=m}$)

The action of Frobenius is defined in [1], §11. A more natural definition is given in [2], but I don't expect it to make computations easier. The tangential base point required (tangent vector 1 at 0) is explained in [1], §15. A rather unsatisfactory computation of the $\zeta^{(p)}(s)$ is essentially the content of [1], 19.6, 19.7, and amounts to $\zeta^{(p)}(s)$ being the following regularization of

$$-p^s \sum_{p \nmid n} \frac{1}{n^s} \quad :$$

write $\sum'$ for a sum extended only to n prime to p and for ℓ prime to p write formally

$$\ell^{1-s} {\sum}' \frac{1}{n^s} = \ell {\sum}' \frac{1}{(\ell n)^s} = \sum_{\alpha^\ell = 1} {\sum}' \frac{\alpha^n}{n^s}\ , \quad \text{hence}$$

$$ {\sum}' \frac{1}{n^s} = (\ell^{1-s} - 1)^{-1} \sum_{\alpha^\ell=1, \alpha\neq 1} \ \sum_{p \nmid n}{}' \frac{\alpha^n}{n^s}.$$

It remains to regularize $\sum' \frac{z^n}{n^s}$ for z not congruent to $1 \mod p$. Let $(n \mod p^N)$ denote the residue of $n \mod p^N$. Define

$$\sum{}' \frac{z^n}{n^s} = \lim_N \sum{}' \frac{z^n}{(n \mod p^N)^s} = \lim_N \sum_k z^{p^N \cdot k} \sum_1^{p^N}{}' \frac{z^n}{n^s}$$
$$:= \lim_N (1 - z^{p^N})^{-1} \sum_1^{p^N}{}' \frac{z^n}{n^s}.$$

[1] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, in: Galois Groups over $\mathbb{Q}$, MSRI Publ. **16** (1989), p. 79–297.

[2] B. Chiarellotto and B. Le Stum, *F-isocristaux unipotents*, Comp. Math. **116** (1999), p. 81–110.

Additional comment:

The problem I ask is something which, using results of Voevodsky as a black box, I can prove is true (for multizeta in general, not just the case asked). I think it would be instructive (and could lead to a better understanding of relations between classical multidzeta) to have a direct proof. I tried to ask the simplest case, but some variants more complicated to formulate, involving roots of 1, could be easier to handle and as instructive (cf. the fact that zeta has a pole at 1, but Dirichlet L-functions don't).

The relation with "periods" is that multizeta are periods, attached to some motive, and crystalline Frobenius are another way to look at the same motive. The same group should control the algebraic relations between periods, and where crystalline Frobenius sits.