Student Projects

We know a certain amount about elliptic curves over $\mathbf{Q}$. All of them admit modular parametrizations– i.e., as quotients of the modular curves;–we know some fragments of the classical Birch-Swinnerton-Dyer Conjecture for them, or at least for those whose analytic Mordell-Weil rank is ≤ 1;– we have (thanks to Kolyvagin, Rubin) some partial results regarding the p-adic "main conjecture" (relative to their *p-anti-cyclotomic* p-adic L-functions);– more recently we have (thanks to Kato) some partial results regarding the p-adic "main conjecture" (relative to their *p-cyclotomic* p-adic L-functions).

The main machine that has been responsible for much of this "knowledge" is Kolyvagin's technique for bounding Selmer groups, by constructing *systematic collections* of global cohomology classes. Now there are four distinct aspects to any of this work, which we can list "backwards".

1. Showing that *systematic collections* of global cohomology classes bound Selmer groups.

2. Constructing these *systematic collections* of global cohomology classes from *motivic objects* (e.g., from *Heegner points* in the Mordell-Weil group of the elliptic curve as in the work connected to the anti-cyclotomic p-adic L functions, or from *Beilinson elements* in algebraic K-theory of the elliptic curve as in the work connected to the cyclotomic p-adic L functions).

3. Constructing the motivic objects.

4. Connecting the motivic objects to a classical L-function.

Of course it is via these four steps that one sees that a classical L function attached to the elliptic curve "controls" some aspect of the arithmetic of the elliptic curve. Clearly this is a somewhat intimidating machine. Nevertheless it is one of the "standard techniques" of our field, and is destined to remain so, and to become more general, if not more streamlined, in the coming years. It seems to us that the relatively painless way you can gain some familiarity with this machine is to do three things at once:

a. Keep the full program in the back of your mind, even if many of the steps are "black boxes".

b. Work on one step at a time, but do this in *considerable detail.*

c. Have a single concrete application in mind as a goal.

Tom Weston and I will be running the project in the Winter School whose aim is, quite simply, to "become familiar with the above machine". In our lectures, Tom Weston and I will do two things. First we will be giving an overview of the above four-step program. Then we will concentrate on a few very particular aspects of the machine (I will try to explain some recent work of myself and Karl Rubin which deals with step 1). As a student project, we suggest taking as "single concrete application" of the above machine the proof that the Shafarevich-Tate group of the elliptic curve $E = X_0(11)$ is trivial. More relevant for an understanding of how the technique works is to aim to prove the weaker result that

- for all but a finite set of primes p, the p-primary part of the Shafarevich-Tate group of the elliptic curve $E = X_0(11)$ is trivial.

Now there are two possible routes to prove this (via Heegner points or via Kato's method using algebraic K-theory). This will be a choice that the group of students should come to collectively. Even beyond this choice, in your presentation(s) you will have to choose only a few aspects of the proof, but the challenge will be to articulate clearly where they fit in in the general program of the proof, and to explain them clearly and give real details. For references:

1. I have given a course (1997) on roughly this material, and have conducted a graduate-student-run seminar last year, again on roughly this material. The notes to the course, and the notes of the seminar, and a link to Karl Rubin's book "Euler Systems" have been conglomerated in http://www.math.harvard.edu/˜weston/mazur.html

2. Rubin, K. : *Euler Systems,* Annals of Math Studies, Princeton University Press, 2000

3. Rubin, K. : Euler Systems and modular elliptic curves, pp. 351-368 in *Galois representations in arithmetic algebraic geometry,* (Ed. A.J. Scholl, R.L. Taylor) Cambridge University Press, 1998

4. Scholl, A.J. : An Introduction to Kato's Euler System, pp. 379-460 in *Galois representations in arithmetic algebraic geometry,* (Ed. A.J. Scholl, R.L. Taylor) Cambridge University Press, 1998