Buzzard's Group Presentation

March 14, 2001

Let $p = 2$. Consider the modular curve $X_0(2) = \left({}_{\Gamma_0(2)}\backslash^{\mathcal{H}}\right)^*$ which parametrizes pairs (E, C) where C is a subgroup of order 2. As an affine curve, $X_0(2)$ is given by $xy = 2^{12}$. And we define

$$f := 1/x = \frac{\Delta(q^2)}{\Delta(q)} = q\prod_{n=1}^{\infty}(1+q^n)^{24}$$

Since $\Delta(q), \Delta(q^2)$ are weight 12 level 2 modular forms, f is a modular function of weight 0 level 2. We have a cannonical map of degree 3 between $X_0(2)$ and $X_0(1)$, which maps the two cusps $0, \infty$ of $X_0(2)$ to the one cusp ∞ of $X_0(1)$ and preserves q-expansions. (Picture was drawn here during the talk.) Since $\Delta(q)$ has a simple zero at ∞ and a double zero at 0 on $X_0(2)$ and $\Delta(q^2)$ has a double zero at ∞ and a single zero at 0 on $X_0(2)$, f has a simple zero at ∞ and a simple pole at 0. Therefore,

$$f : X_0(2) \xrightarrow{\sim} \mathbb{P}^1$$

Now if we consider $X_0(2)$ as a rigid analytic space, the collection $\{1, f, f^2, f^3, \dots\}$ forms a basis of the space of funtions on $\{x \in X_0(2) \;\;:\;\; |f(x)| \le 1\}$ such that $\sum_{n=0}^{\infty} a_n T^n$, $|a_n|_2 \to 0$ as $n \to \infty$, and where $|T| < 1$. The reduction of $X_0(2)_{\mathbb{Q}_2}$ to $X_0(2)_{\mathbb{F}_2}$ takes an 2-adic annulus of supersingular points $2^{-12} > |x| > 1$ to the singular point of $xy = 0$. (Picture could be drawn here.)

If we write $f(q) = \sum_{n=1}^{\infty} a_n q^n$, then our goal is to compute what U_2 looks like on $\{1, f, f^2, f^3, \dots\}$. If we let $\alpha = f(\sqrt{q})$ and $\beta = f(-\sqrt{q})$, then we have

$$\begin{aligned}
\alpha + \beta &= \sum a_n q^{n/2} + \sum (-1)^n a_n q^{n/2} \\
&= 2\sum a_n q^{n/2} + 2\sum a_{2n} q^n \\
&= 2U_2 f
\end{aligned}$$

Therefore $U_2 f = \frac{1}{2}(\alpha + \beta)$, and $U_2 f : X_0(2) \longrightarrow \mathbb{P}^1_{\mathbb{Q}_2}$ is a map of degree 2.

If $f(q)$ is a modular form for Γ, then $f(\delta q)$ is a modular form for $\delta^{-1}\Gamma\delta$. So $f(\sqrt{q}) = f(\tau/2) = f(\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)\tau)$ is a modular form on $\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)^{-1}\Gamma_0(2)\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right) \supseteq \Gamma(2) = \left\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \bmod 2\right\}$

Note that we know the degree of α, β are 2, and looking at where our maps factor through we have

$$\begin{tikzcd}
\alpha,\ \beta:\ X(2) \arrow[rr, "2"] \arrow[dr, "2", "\tau\mapsto\tau/2"'] & & \mathbb{P}^1 \\
& X_0(2) \arrow[ur, "1", "f"'] &
\end{tikzcd}$$

Thus $\deg(U_2 f) \leq 4$ on $X(2)$ and we have the following diagram

$$\begin{tikzcd}
U_2 f = 1/2(\alpha+\beta) : X(2) \arrow[rr, "\leq 4"] \arrow[dr, "2"'] & & \mathbb{P}^1 \\
& X_0(2) \arrow[ur, "\leq 2"'] &
\end{tikzcd}$$

So $U_2 f$ will factor through $X_0(2)$. Now the question becomes how far out do we need to look for f expansions?

By the above discussion we see that $\deg(U_2 f) = 2$ on $X_0(2)$, so

$$U_2 f = \frac{\_ + \_ f + \_ f^2}{\_ + \_ f + \_ f^2}$$

and we need to figure out what should go in the blanks. Since $f : 0 \to \infty$ and $f : \infty \to 0$, $U_2 f = \_ + \_ f + \_ f^2$. By a computer calculation we see that $U_2 f = 24 f + 2084 f^2$.

The matrix for $2U_2$ with respect to $\{1, f, f^2, f^3, \dots\}$ will be denoted by (c_{ij}), where $2U_2 f^j = \sum_i c_{ij} f^i$. And this matrix actually begins

$$\begin{pmatrix}
2 & 0 & 0 & \longrightarrow \\
0 & 48 & 2 & \\
0 & 2^{12} & 2304 & \dots \\
0 & 0 & \vdots & \\
\downarrow & \downarrow & \vdots &
\end{pmatrix}$$

So we are interested in a generating function. Let $g = \sum b_n q^n$, then $2U_2 g = 2\sum b_{2n} q^n = g(\sqrt{q}) + g(-\sqrt{q})$. So if $g = f^j$, $2u_2 f^j = \alpha^j + \beta^j = \sum c_{ij} f^i$. Now

$$\begin{aligned}
\sum (\alpha y)^j + (\beta y)^j &= \sum c_{ij} f^i y^j \\
&= \frac{1}{1-\alpha y} + \frac{1}{1-\beta y} \\
&= \frac{2 - (\alpha+\beta) y}{1 - (\alpha+\beta) y + \alpha\beta y^2}
\end{aligned}$$

In fact

$$
\begin{aligned}
\alpha\beta &= \sqrt{q}\prod_n (1+(\sqrt{q})^n)^{24} \cdot -\sqrt{q}\prod_n (1+(-\sqrt{q})^n)^{24} \\
&= -q \prod_{n \text{ even}} \left((1+q^n)^2\right)^{24} \cdot \prod_{n \text{ odd}} (1-q^n)^{24} \\
&= -q \prod_n \frac{\left((1+q^n)^2\right)^{24}(1-q^n)^{24}}{(1-q^{2n})^{24}} \\
&= -q \prod_n (1+q^n)^{24} \\
&= -f
\end{aligned}
$$

So we have a generating function

$$\frac{2-(48f+2^{12}f^2)y}{1-(48f+2^{12}f^2)y-fy^2}$$

Can you figure out the c_{ij} from this generating function?

In order to make our argument rigorous, we need to show that $2U_2$ is a compact operator on the ρ-overconvergent modular forms with $|\rho|_2 < 1$. We know that if $M = (m_{ij})$ is an infinite matrix giving an operator, then M is compact if and only if for $\gamma_i = \sup_j |m_{ij}|_2$, we have that $\gamma_i \to 0$ as $i \to \infty$. The problem we face is that $c_{i,2i} = 2$, so we do not have a compact operator for the basis $\{1, f, f^2, f^3, \dots\}$.

Since we do not have a compact operator on the space of all 2-adic modular forms, we must restrict to ρ-overconvergent modular forms. Pick $\omega \in \overline{\mathbb{Q}}_2$ such that $|\omega|_2 < 1$. We now adjust our basis: $\{1, (\omega f), (\omega f)^2, \dots\}$ and look at $2U_2$ as a matrix with respect to this new basis

$$
\begin{aligned}
2U_2(\omega f)^i &= \omega^j \sum_i c_{ij} f^i \\
&= \sum_i c_{ij}\omega^{j-i}(\omega f)^i
\end{aligned}
$$

So in terms of the new basis $2U_2 = (d_{ij}) = (c_{ij}\omega^{j-i})$. Now letting $\omega = 2^l$ with l a positive rational and recalling the generating function for (c_{ij}), we have

$$
\begin{aligned}
\sum d_{ij} f^i y^j &= \sum c_{ij}\omega^{j-i}(\omega y)^j \\
&= \frac{2-(2^4\cdot 3f+2^{12}f^2\omega^{-1})y}{1-(2^4\cdot 3f+2^{12}f^2\omega^{-1})y-\omega f y^2} \\
&= \frac{2-(2^4\cdot 3f+2^{12-l}f^2)y}{1-(2^4\cdot 3f+2^{12-l}f^2)y-2^l f y^2}
\end{aligned}
$$

which has a power series in $\mathcal{O}_{\overline{\mathbb{Q}}_2}$. Thus $|d_{ij}|_2 < |2^{il}|_2 = 2^{-il} \to 0$ as $i \to 0$. So $2U_2$ is compact on $\{1, (\omega f), (\omega f)^2, \dots\}$ as desired.

We have been interested in computing (c_{ij}) and a result of Frank Calegari shows this matrix is computable for U_2.

Theorem. $c_{ij} = \dfrac{2^{8i-4j}3j(i+j-1)!}{(2i-j)!(2j-i)!}$ where $c_{ij} = 0$ whenever not well-defined.

Proof. There is a recurrence relation

$$\begin{aligned} F_1 &= 2U_2 f = \alpha + \beta = 48f + 4096f^2 \\ F_n &= 2U_2 f^n = \alpha^n + \beta^n \\ F_{n+1} &= 2U_2 f^{n+1} = \alpha^{n+1} + \beta^{n+1} = (\alpha+\beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}) \end{aligned}$$

then

$$\begin{aligned} F_{n+1} &= 2U_2 f \cdot 2U_2 f^n + f(2U_2)(f^{n-1}) \\ &= (48f + 4096f^2)F_n + fF_{n-1} \end{aligned}$$

so $c_{i+1,j+1} = 48c_{i,j} + 4096c_{i-1,j} + c_{i,j-1}$ and the rest follows by an ugly calculation.

Ideally we would like to have a closed form, which we hope to derive from

$$\frac{2 - (\alpha+\beta)y}{1 - (\alpha+\beta)y + \alpha\beta y^2}$$

but what we have been able to compute is that

$$c_{ij} = 2^{8i-4j}3^{2j-1}\left(\left(\begin{array}{c} j \\ 2j-i \end{array}\right) + j\sum_{k=1}^{[\frac{2j-i}{3}]} \frac{3^{-3k}}{k}\left(\begin{array}{c} j-(k+1) \\ j-2k \end{array}\right)\left(\begin{array}{c} j-2k \\ 2j-i-3k \end{array}\right)\right)$$

but we have been unable to reduce this to Frank's result.

There are many related open and fun problems. We can try the same calculations with $E_2(q) =$ weight 2, level 2 Eisenstein series, then $E_2(q)$, $E_2(q^2)$ are overconvergent modular forms of weight 2 and level 1. Let $g = \dfrac{E_2(q)/E_2(q^2) - 1}{24}$ (which is overconvergent of weight 0 and level 2) try to compute the matrix in terms of the paramaters

$$\begin{aligned} U_2(g) &= 0 \\ U_2(g^2) &= \sum\left((2i+1)3^{i-2}2^{3(j-1)}(-1)^{i+1}\right)g^i \\ U_2(g^3) &= 0 \\ U_2(g^4) &= \pm\left(U_2(g^2)\right)^2 \end{aligned}$$

There are many interesting problems for $p = 3, 5, 7, 13, \ldots$. What are nice paramaters ... try to find nice formulae.