P-ADIC MODULAR FORMS BY KEVIN BUZZARD - LECTURE 1

1. History

Serre and Katz (1972): Understand congruences mod p and mod p^n between modular forms (see Antwerp).
Serre (1972): Application to p-adic L-functions.
Katz (1972): Wanted conceptual explanation of Atkin's work and Serre's paper.
Hida (early 80's): "Ordinary" subspace of space of p-adic modular forms, and constructed families.
Gouvea (1987): Makes conjectures that Mazur's deformation rings correspond to Hida's ring of Hecke operators.
Wiles and Taylor-Wiles (1994): Deformation rings = Hecke rings
Taylor et al: Applications to conjecture of Artin.
Gouvea-Mazur: Conjecture "forms come in big families"
Coleman: Uses rigid analytic approach, proved that forms come in small families.
Wan, Smithline, Emerton (more recently)
Stein (more recently): Enables us to do calculations.

2. Gouvea-Mazur Conjecture

Let p be prime, $N \geq 1$ an integer prime to p and $S_k(\Gamma_0(Np))$ the space of cusp forms of weight k and level Np (think of N, p fixed with k varying). Consider U_p, the Hecke operator, acting on $S_k(\Gamma_0(Np))$, the characteristic polynomial of U_p lies in $\mathbb{Z}[X]$, think of the roots, $\lambda_1, ..., \lambda_n$ of this polynomial as elements of $\overline{\mathbb{Q}}_p$ so that they have valuations. For $\alpha \in \mathbb{Q}$ define

$$d(k, \alpha) = \#\{\lambda_i : v(\lambda_i) = \alpha\}$$

Conjecture 1. *Let $k_1, k_2 \in \mathbb{Z}$, $\alpha \in \mathbb{Q}$, $\alpha \geq 0$ and assume that*
(i) $k_1, k_2 > \alpha + 1$
(ii) $k_1 \equiv k_2 \bmod (p-1)p^r$ for some $r \in \mathbb{Z}, r \geq \alpha$
then $d(k_1, \alpha) = d(k_2, \alpha)$.

One approach:
Remove hypothesis (i) but instead let $d(k, \alpha)$ be the number of eigenvalues, with valuation α, of U_p acting on huge space of "overconvergent p-adic modular forms of weight k".

Classical mod forms of wt $k \subset$ Overconvergent p-adic mod forms of wt k

3. Modular Forms

Definition 1. *A modular form of level 1 and weight $k \in \mathbb{Z}$ is an analytic function $f : \mathcal{H} \to \mathbb{C}$ such that*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and f satisfies boundedness conditions.

If $k = 0$ the only modular forms are constants. One could weaken the boundedness conditions by, e.g., allowing f to have some poles, and then one gets a whole lot of interesting functions (e.g. j, which has a "pole at infinity"). For $k = 0$ a modular form is a function on $SL_2(\mathbb{Z}) \setminus \mathcal{H}$, however if $k \neq 0$ then one can't think of a modular form as a function on $SL_2(\mathbb{Z}) \setminus \mathcal{H}$ because of the $(c\tau + d)^k$ factor.

$SL_2(\mathbb{Z}) \setminus \mathcal{H}$ is a parameter space for elliptic curves:
$\tau \in \mathcal{H} : \mathbb{C}/\langle 1, \tau \rangle$ is a 1-dimensional complex torus.

$$\begin{aligned} \mathbb{C}/\langle 1, \tau \rangle \cong \mathbb{C}/\langle 1, \sigma \rangle &\iff \lambda \langle 1, \sigma \rangle = \langle 1, \tau \rangle \text{ for some } \lambda \in \mathbb{C} \\ &\iff \lambda = c\tau + d, \sigma\lambda = a\tau + b \\ &\iff \sigma = \frac{a\tau + b}{c\tau + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \end{aligned}$$

So let's remember λ by considering functions f sending a pair (E, ω) to a complex number $f(E, \omega)$ where E is a complex 1-dimensional torus and ω is a non-vanishing global differential such that $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$ A modular form of weight k is an analytic rule sending (E, τ) to $f(E, \tau)$ such that $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$ and f satisfies boundedness conditions.

That's "encoded" the functional equation, now we wish to encode the word "analytic" by allowing families of tori.

If S is a base complex manifold and $\pi : T \to S$ is a family of tori then we should try and make sense of $f(T/S, \omega)$ where ω is now a family of non-vanishing differentials: $f(T/S, \omega)$ should be an analytic function $S \to \mathbb{C}$ and if $s \in \mathbb{C}$ then $f(T/S, \omega)(s) = f(T_s, \omega_s)$.

This is now getting very non-computable but also much more algebraic.

Definition 2. *Let R_0 be a ring (commutative with identity). A modular form of level 1 and weight k defined over R_0 is a rule, which to every pair $(E/R, \omega)$ where,*
(1) R is an R_0-algebra
(2) E/R is an elliptic curve, $\pi : E \to Spec(R)$
(3) $\omega \in H^0(E, \Omega^1)$ is a nowhere vanishing differential
an element $f(E/R, \omega) \in R$ such that

(a) $f(E/R,\omega)$ only depends on the isomorphism class of the data
(b) if E/R_1 and E'/R_2 are elliptic curves, and we have $\beta : Spec(R_2) \to Spec(R_1)$, then we can form the pullback β^*E of E to $Spec(R_2)$ along β, and if we have an isomorphism from E' to β^*E then the data of the f's should match up too.
(c) $f(Tate(q),\omega_{can}) \in R_0[[q]]$ (boundedness conditions)
(d) $f(E/R,\lambda\omega) = \lambda^{-k} f(E/R,\omega)$ for all $\lambda \in R^*$

Explanation of (c): There is an elliptic curve called $Tate(q)$ defined over the p-adic completion of $R_0((q))$, with a canonical nowhere-vanishing differential ω_{can}, and by definition $f(Tate(q),\omega_{can})$ will be in the p-adic completion of $R_0((q))$, and the assertion is that it has to be in the much smaller ring $R_0[[q]]$ (the point is that we want to rule out poles at infinity).

Non-computable definition: a modular form is a well behaved rule on (Elliptic curves, differentials).

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