OUTLINE OF A COURSE ON ELLIPTIC CURVES AND GROSS-ZAGIER THEOREMS OVER FUNCTION FIELDS

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1. Review of elliptic curves over function fields

- Definitions and examples. Constant, isotrivial, and non-constant curves.
- The Mordell-Weil theorem.
- Constant curves. The lattices of Elkies, Shioda, et. al.
- Torsion is uniformly bounded. Ranks are unbounded.
- L-functions.
- Grothendieck's analysis of L-functions gives analytic continuation, functional equation.
- L-functions should be viewed as functions of characters of the idèle class group.
- Zarhin's theorem: $\chi \mapsto L(E, \chi)$ determines E up to isogeny.
- The conjecture of Birch and Swinnerton-Dyer.
- Work of Tate and Milne: $\mathrm{ord}_{s=1} L(E, s) \geq \mathrm{Rank}_{\mathbb{Z}} E(F)$ with equality if and only if Ш is finite.
- Outline of the proof:
  - The elliptic surface $\mathcal{E}/\mathbf{F}_q$ corresponding to E/F.
  - $L(E, s) = \det(1 - q^{-s}\mathrm{Fr}|H)$ for a certain $H \subseteq H^2(\overline{\mathcal{E}}, \mathbf{Q}_\ell)$
  - Points on E correspond to curves on $\mathcal{E}$. Heights are essentially intersection numbers.
  - Cycle classes of curves give rise to zeroes of the L-function.
  - Finiteness of Ш $\Leftrightarrow$ weak BSD comes from the Kummer sequence on $\mathcal{E}$ and Ш $= Br$.
- Other work: Brown, Rück-Tipp, Longhi, Pàl.
- References: [Gross], [Zarhin], [Groth], [Milne80], [Tate66], [Milne75], [C-Z], Gross in [Storrs].

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2. Automorphic forms and analytic modularity

- Additive characters, multiplicative characters, conductors and real parts.
- Definition of $\mathcal{A}(K,\phi)$, automorphic forms of level K and central character ϕ.
- Analogue with functions on upper half plane. The double coset space X where automorphic forms live.
- X is the set of isomorphism classes of rank 2 vector bundles with level structure (up to twisting by a line bundle).
- Structure of X. (Riemann-Roch and stability.)
- Petersson inner product.
- Cusp forms.
- Hecke operators, new and old forms.
- Fourier expansions.
- L-functions.
- Functional equations.
- Harmonic forms.
- Constructions of forms, classically and in terms of vector bundles:
  - Eisenstein series
  - Poincaré series
  - Theta functions
  - Converse theorems
  - Deligne's theorem: there is a form f such that $L(E,s) = L(f,s)$
  - Drinfeld's geometric Langlands construction
- Interesting linear functionals on $\mathcal{A}(K,\phi)$ are represented as PIP with interesting forms $f \in \mathcal{A}(K,\phi)$.
- Half of the Gross-Zagier computation is to find the Fourier expansion of the form representing $f \mapsto L'_K(f,1)$.
- References: [Weil], [Serre], [Gek], [Del], [Drin83].

3. Drinfeld modular curves and geometric modularity

- The ring A of functions regular outside ∞
- For k of characteristic p, $\mathrm{End}_k(\mathbf{G}_a)$ is the twisted polynomial ring $k\{\tau\}$, $\tau a = a^p \tau$.
- Definition of Drinfeld modules. Rank, characteristic, height.
- Examples.
- Morphisms.
- Division points.
- Isogenies.
- Endomorphisms.
- Complex multiplication.
- Level structures.
- Modular curves.
- Analytic description of Drinfeld modular curves.
- The adelic version of the analytic description.
- The building map.
- Drinfeld reciprocity: relating the cohomology of the modular curve to automorphic forms.
- Geometric modularity via Drinfeld reciprocity, Deligne + converse theorems, and Zarhin.
- References: [Drin74], [D-H], [G-R], [AB], [Ohio].

4. Overview of the Gross-Zagier computation and application to elliptic curves

- Heegner points on $X_0(\mathfrak{n})$: existence, construction, Galois action. The Heegner point $P_K \in J_0(\mathfrak{n})(K)$.
- Goal: $L'_K(f,1) = c\,\mathrm{ht}(P_{K,f})$ (c a non-zero constant) for new eigenforms $f \in \mathcal{A}(\Gamma_0(\mathfrak{n}\infty), |\cdot|^{-2})$.
- Key reduction: do it for all f at once.
  - Let h_{an} be the form such that $(f, h_{an})_{PIP} = L'_K(f,1)$.
  - Let h_{alg} be the form with Fourier coefficients $\langle P_K, T_{\mathfrak{m}} P_K\rangle_{ht}$.
  - A formal Hecke algebra argument shows that the goal is equivalent to the equality $h_{an} = c\,h_{ht}$. Prove this coefficient by coefficient.
- The analytic computation.
  - Rankin's method shows that $L_K(f,s) = (f, h_s)_{PIP}$ where h_s is the product of a CM form (theta series) and an Eisenstein series which is a function of s.
  - Compute a trace to make the level of h_s $\mathfrak{n}\infty$.
  - Take the derivative at $s=1$: $h = \frac{d}{ds}h|_{s=1}$.
  - Do a "harmonic projection": find h_{an} harmonic such that $(f, h_{an})_{PIP} = (f,h)_{PIP}$ for all harmonic forms f.
- The algebraic computation
  - Interpret height as a sum of local intersection numbers.
  - At finite places, intersection number counts the number of isogenies between certain Drinfeld modules x, y over finite rings $\mathcal{O}_v/(\pi_v^n)$. (Use the moduli interpretation of points.)
  - Count these isogenies using the ideal theory of the quaternion ring $\mathrm{End}(x)$.
  - At ∞ there is no convenient moduli interpretation. Compute the local height using a Green's function, exactly as in the original G-Z. This is a very analytic way to calculate a rational number, but it meshes well with analytic aspects of the harmonic projection calculation.
- Application to elliptic curves. Show $\mathrm{ord}_{s=1} L(E,s) \le 1 \Rightarrow$ BSD for E/F by using G-Z formula and non-vanishing results for L-functions. In function field case, non-vanishing results are used for some useful preliminary reductions, and to find a good K/F.
- References: [G-Z].

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