JACOBIANS OF GENUS 1 CURVES

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The problem

C: a genus 1 curve defined over $\mathbb{Q}$

J: the Jacobian curve of C

Here we consider the following problem:

Given C, find J.

We present a unified treatment of each of the following cases:

- a double cover of $\mathbb{P}^1$
- a cubic plane curve
- an intersection of quadric surfaces in $\mathbb{P}^3$

The work of Weil

In his 1954 paper *Remarques sur un mémoire d'Hermite*, A. Weil produced the following formulas:

If C is given by

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$$

then J can be written in Weierstrass form as

$$\zeta^2 = 4\xi^3 - i\xi - j$$

where

$$\begin{aligned} i &= a_0a_4 - 4a_1a_3 + 3{a_2}^2 \\ j &= a_0a_2a_4 + 2a_1a_2a_3 - a_0{a_3}^2 - a_4{a_1}^2 - {a_2}^3. \end{aligned}$$

Invariant theory

$\mathcal{F}_{n,m}$: forms of degree n in m variables

Let F be such a form (or a collection of forms).

The GL_m action on $\mathcal{F}_{n,m}$ gives rise to the notions of invariants and covariants of F.

An invariant of F is a polynomial expression in the coefficients of F.

A covariant is another form in m variables, whose coefficients are polynomial expressions in the coefficients of F. In particular, an invariant is a covariant of degree zero.

A dependence relation among covariants is called a syzygy.

Connection to curves

It is via these fundamental syzygies that we obtain Weierstrass models for the Jacobians of our curves.

Namely,

- for double covers of $\mathbb{P}^1$, we use a syzygy satisfied by covariants of a binary quartic form.

- for plane cubics, we use a syzygy satisfied by covariants of a ternary cubic form.

- for space quartics, we use a syzygy satisfied by covariants of a pair of quaternary quadratic forms.

Ternary cubic forms

The general plane cubic is given by the equation $U = 0$, where U is a ternary cubic form:

$$U = ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y + 6mxyz.$$

Invariants of U: S, T
Covariants of U: U, H, Θ, J.

They satisfy the following syzygy:

$$\begin{aligned} J^2 = 4\Theta^3 &+ 108S\Theta H^4 - 27TH^6 + TU^2\Theta^2 \\ &- 4S^3U^4\Theta + 2STU^3\Theta H - 72S^2U^2\Theta H^2 \\ &- 18TU\Theta H^3 - 16S^4U^5H - 11S^2TU^4H^2 \\ &- 4T^2U^3H^3 + 54STU^2H^4 - 432S^2UH^5. \end{aligned}$$

or

$$J^2 = 4\Theta^3 + 108SH^4\Theta - 27TH^6.$$

Jacobian of a plane cubic

$$J^2 = 4\Theta^3 + 108SH^4\Theta - 27TH^6$$

Dividing through by H^6 gives

$$\left(\frac{J}{H^3}\right)^2 = 4\left(\frac{\Theta}{H^2}\right)^3 + 108S\left(\frac{\Theta}{H^2}\right) - 27T.$$

So if we let E denote the curve

$$\zeta^2 = 4\xi^3 + 108S\xi - 27T,$$

then the map

$$\phi\colon P \mapsto \left(\frac{\Theta(P)}{H^2(P)}, \frac{J(P)}{H^3(P)}\right)$$

is a rational map from C to E.

Two quadratic forms: A space quartic

Let U and V denote quadratic forms in four variables. The covariant theory of such a system is also well known.

Five basic invariants: $\sigma_0, \ldots, \sigma_4$,

Five basic covariants: U, V, F_1, F_2, G.

These satisfy a fundamental syzygy:

$$G^2 = \sigma_0 {F_1}^4 - \sigma_1 {F_1}^3 F_2 + \sigma_2 {F_1}^2 {F_2}^2 - \sigma_3 F_1 {F_2}^3 + \sigma_4 {F_2}^4,$$

or

$$\left(\frac{G}{{F_2}^2}\right)^2 = \sigma_0 \left(\frac{F_1}{F_2}\right)^4 - \sigma_1 \left(\frac{F_1}{F_2}\right)^3 + \sigma_2 \left(\frac{F_1}{F_2}\right)^2 - \sigma_3 \frac{F_1}{F_2} + \sigma_4.$$

Let C' denote the curve

$$y^2 = \sigma_0 x^4 + \sigma_1 x^3 + \sigma_2 x^2 + \sigma_3 x + \sigma_4.$$

Then the map

$$\phi \colon P \mapsto \Big(\frac{-F_1(P)}{F_2(P)}, \frac{G(P)}{F_2(P)^2}\Big)$$

is a rational map from C to C'. Composing this with Weil's result gives a rational map from C to a Weierstrass model.

In each case the covariant theory of the right type of object provided us with rational maps from our curves to Weierstrass models of elliptic curves. We want to conclude that in each case the elliptic curve is in fact a model of the Jacobian.

A lemma

An n-torsion packet on C is a set of n^2 points such that for any pair, say P and Q, the divisor class $[n(P-Q)]$ is trivial; that is, $[P-Q] \in J[n]$.

LEMMA Let C be a genus 1 curve defined over $\mathbb{Q}$, and J its Jacobian. Let $W \subset \mathbb{P}^2$ be a Weierstrass model. If there is a morphism $\phi\colon C \to W$ defined over $\mathbb{Q}$, so that $\phi^{-1}(\infty)$ is an n-torsion packet, then $J \simeq W$, the isomorphism being defined over $\mathbb{Q}$.

REMAINING QUESTION: In each case, what is $\phi^{-1}(\infty)$?

Plane cubic case

C: a cubic plane curve $U = 0$

W: $\zeta^2 = 4\xi^3 + 108S\xi - 27T$

ϕ: $P \mapsto \left(\frac{\Theta(P)}{H^2(P)}, \frac{J(P)}{H^3(P)}\right)$

or projectively

ϕ: $P \mapsto [\Theta(P)H(P) : J(P) : H^3(P)]$

So, $\phi^{-1}(\infty) = \phi^{-1}[0 : 1 : 0]$ is precisely where H vanishes on C.

H is the Hessian covariant of U, so H vanishes at the nine flex points of C.

CLAIM: The nine flex points form a 3-torsion packet.

Let P and Q be two flex points, with tangent lines L_1 and L_2. Then the divisor $(3P - 3Q)$ is the divisor of the function $\frac{L_1}{L_2}$.