Programme for the Tucson meeting

The local-global principle

Let X/k be a smooth, projective variety over a number field k. Let k_v denote the completion of k at a place v. Let $X(k)$ denote the set of k-rational points of X and $X(k_v)$ the set of k_v-rational points, equipped with the topology induced by the local field k_v.

One may ask whether the diagonal embedding

$$X(k) \hookrightarrow \prod_{\mathrm{all} v} X(k_v)$$

has dense image (weak approximation). Note that when this is the case, from $X(k_v) \neq \emptyset$ for all v one deduces $X(k) \neq \emptyset$ (Hasse principle).

The aim of this session is to explore the extent to which these properties hold, or at any rate to control their failure. The three main techniques involved are:

1) *The fibration method*

2) *Descent* (study of torsors under groups of multiplicative type)

3) Use of the *Brauer group* of (possibly open) varieties

There are strong interactions between these three techniques.

Each of the main speakers (J.-L. Colliot-Thélène, A. N. Skorobogatov, D. Harari) will give a one hour talk, the three talks being coordinated.

Surveys, books.

J.-L. Colliot-Thélène et J.-J. Sansuc, La descente sur les variétés rationnelles. In Journées de géométrie algébrique d'Angers (Juillet 1979), édité par A.Beauville, Sijthof and Noordhof (1980) 223-237.

Yu. I. Manin, *Cubic forms : algebra, geometry, arithmetic*, Second edition, North Holland, Amsterdam, 1986

Yu. I. Manin and M. A. Tsfasman, Rational varieties: algebra, geometry, arithmetic, Uspekhi Mat. Nauk **41** (1986) 43-94 (=Russian math. surveys **41** (1986) 51-116)

J.-J. Sansuc, Principe de Hasse, surfaces cubiques et intersections de deux quadriques, in *Journées arithmétiques de Besançon 1985*, Astérisque **147-148** (1987) 183-207

J.-L. Colliot-Thélène, Arithmétique des variétés rationnelles et problèmes birationnels. Proceedings of the International Congress of Mathematicians, Berkeley, California 1986, (1987), Tome I, 641-653.

J.-L. Colliot-Thélène, L'arithmétique des variétés rationnelles, Annales de la Faculté des sciences de Toulouse, vol. **I**, numéro 3 (1992) 295-336.

J.-L. Colliot-Thélène, The Hasse principle in a pencil of algebraic varieties, Proceedings of the Tiruchirapalli conference (India, January 1996), ed. M. Waldschmidt and K. Murty, Contemporary Mathematics, **210** (1998) 19-39.

A more difficult paper :

J.-L. Colliot-Thélène et J.-J. Sansuc, La descente sur les variétés rationnelles, II, Duke Math. J. **54** (1987) 375-492.

Topics to be presented by students

Six 30min talks:

Talks 1 to 4 will be given before the main lectures, talks 5 and 6 after the main lectures.

1) The Picard group (Andrew Nestler/Matei Stroila)

2) The Brauer group (Adrian Barbu)

The Leray spectral sequence for $X \to Spec(k)$ and the sheaf $\mathbf{G}_m$.

Computation of the Picard group of a conic bundle surface over $\mathbf{P^1}$.

Cover sections 1 and 2 of the following paper.

J.-L. Colliot-Thélène and P. Swinnerton-Dyer, Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties J. für die reine und ang. Mathematik (Crelle) **453** (1994) 49-112.

Further reading :

A. Grothendieck, Le groupe de Brauer II , in *Dix exposés sur la cohomologie des schémas*, Masson & Cie, Paris, North-Holland, Amsterdam, 1968.

Some aspects of Chapter 4 in Milne's Étale Cohomology (do not insist on the Azumaya aspect).

Given X/k a variety over a local field, and A in the cohomological Brauer group $Br(X) = H^2_{et}(X, \mathbf{G}_m)$, show that the map $X(k) \to Br(k)$ sending P to $A(P)$ is locally constant.

3) Arithmetic, class field theory (Marko Petzold)

A summary of the results of class field theory. The basic exact sequence

$$0 \to \mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z} \to 0.$$

Possibly the Tate-Nakayama theory.

For K/k cyclic, N the norm map from K to k, injectivity of the map

$$k^*/NK^* \to \oplus_v k_v^*/N(K \otimes_k k_v)^*.$$

Possibly a mention of Tchebotarev's theorem.

References :

Algebraic number theory, J. W. S. Cassels and A. Fröhlich ed., Academic Press (London, 1967).

S. Lang, Algebraic Number Theory

J. Neukirch, Algebraische Zahlentheorie

4) Counterexamples to the Hasse principle and weak approximation (Nero Budur/Fatih Unlu)

Various counterexamples to the Hasse principle and weak approximation (curves of genus 1, norms for non-cyclic extensions). The simplest one is Iskovskih's counterexample for the Hasse principle (over the rationals) :

$$y^2 + z^2 = (3 - x^2)(x^2 - 2)$$

Swinnerton-Dyer, Two special cubic surfaces, Mathematika 9 (1962) 54-56.

Last section of

J.-L. Colliot-Thélène et J.-J. Sansuc, On the Chow groups of certain rational surfaces : a sequel to a paper of S.Bloch. Duke Math.J. 48 (1981) 421-447.

Example 5.4 in

J.-L. Colliot-Thélène, D. Coray et J.-J. Sansuc, Descente et principe de Hasse pour certaines variétés rationnelles. J.für die reine und ang. Math. (Crelle) Bd. 320 (1980) 150-191.

First paragraphs of :

J.-J. Sansuc, Descente et principe de Hasse pour certaines variétés rationnelles, Séminaire de théorie des nombres, Paris 1980-1981, Progress in Math. **22**, Birkhäuser 1982.

Section 7 of :

J.-L. Colliot-Thélène, D. Coray et J.-J. Sansuc, Descente et principe de Hasse pour certaines variétés rationnelles. J.für die reine und ang. Math. (Crelle) Bd. 320 (1980) 150-191.

Section 15 of :

J.-L. Colliot-Thélène, J.-J. Sansuc and Sir Peter Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces, I, J. für die reine und angew. Math. (Crelle) **373** (1987) 37-107; II, ibid. **374** (1987) 72-168.

Section 8 of :

J.-L. Colliot-Thélène and P. Salberger, Arithmetic on singular cubic hypersurfaces. *Proc. London Math. Soc.* (3) **58** (1989), 519-549.

Section 5 of

D. Harari, Principe de Hasse et approximaiton faible sur certaines hypersurfaces, Annales de la faculté des Sciences de Toulouse, vol. IV (1995) 731-762.

More advanced counterexamples :

D. Harari, Obstruction de Manin "transcendante". Number Theory, Paris 1993-94, Séminaire de théorie des nombres de Paris, ed. S. David, Cambridge University Press 1996

5) Weak approximation for linear algebraic groups (Gautam Chinta)

Proof that the Brauer-Manin obstruction to weak approximation for linear algebraic groups is the only obstruction. This theorem of Sansuc was given a simpler proof via the fibration method.

References

B. Kunyavskiĭ and A. N. Skorobogatov, A criterion for weak approximation on lnear algebraic groups, in Séminaire de théorie des nombres de Paris 1988-89, éd. C. Goldstein, Progr. in Math **91**, Birkhäuser 1991, p. 215-217.

D. Harari, Méthode des fibrations et obstruction de Manin, Duke Math. J. **75** (1994) 221-260. (Theorem 5.3.1)

6) The Cassels-Tate pairing on the Tate-Shafarevich group versus the Brauer-Manin pairing. (Victor Scharaschkin)

Explain how the classical results on curves of genus one (the Cassels-Tate dual exact sequence) translate into the Brauer-Manin obstruction.

References :

Yu. I. Manin, Le groupe de Brauer-Grothendieck en géométrie diophantienne, in Actes Congrès Intern. Math. (Nice 1970) Gauthiers-Villars, Paris 1971, Tome **1** 401-411.

L. Wang, Brauer–Manin obstruction to weak approximation on abelian varieties, Israel Journal of Mathematics **94** (1996) 189-200.

B. Poonen and M. Stoll, The Cassels-Tate pairing for polarized abelian varieties, by Bjorn Poonen and Michael Stoll, Algebraic-Number-Theory/0088/ (Dec. 1997).

J.-L. Colliot-Thélène, Conjectures de type local-global sur l'image de l'application cycle en cohomologie étale, à paraître dans Proceedings of the Seattle conference on Algebraic K-Theory (1997).

Curves in their Jacobians (preprints by Skorobogatov and by Scharaschkin)