CHAPTER 0
PRELIMINARY MATERIAL

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This chapter gives some preliminary material on number theory and algebraic geometry.

Section 1 gives basic preliminary notation, both mathematical and logistical. Section 2 describes what algebraic geometry is assumed of the reader, and gives a few conventions that will be assumed here. Section 3 gives a few more details on the field of definition of a variety. Section 4 does the same as Section 2 for number theory.

The remaining sections of this chapter give slightly longer descriptions of some topics in algebraic geometry that will be needed: Kodaira's lemma in Section 5, and descent in Section 6.

§1. General notation

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the ring of rational integers and the fields of rational numbers, real numbers, and complex numbers, respectively. The symbol $\mathbb{N}$ signifies the natural numbers, which in this book start at zero: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. When it is necessary to refer to the positive integers, we use subscripts: $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$. Similarly, $\mathbb{R}_{\geq 0}$ stands for the set of nonnegative real numbers, etc.

The set of **extended real numbers** is the set $\bar{\mathbb{R}} := \{-\infty\} \coprod \mathbb{R} \coprod \{\infty\}$. It carries the obvious ordering.

If k is a field, then $\bar{k}$ denotes an algebraic closure of k. If $\alpha \in \bar{k}$, then $\operatorname{Irr}_{\alpha,k}(X)$ is the (unique) monic irreducible polynomial $f \in k[X]$ for which $f(\alpha) = 0$.

Unless otherwise specified, the wording **almost all** will mean all but finitely many.

Numbers (such as Section 2, Theorem 2.3, or (2.3.5)) refer to the chapter in which they occur, unless they are preceded by a number or letter and a colon in bold-face type (*e.g.*, Section **3:**2, Theorem **A:**2.5, or (**7:**2.3.5)), in which case they refer to the chapter or appendix indicated by the bold-face number or letter, respectively.

§2. Conventions and required knowledge in algebraic geometry

It is assumed that the reader is familiar with the basics of algebraic geometry as given, *e.g.*, in the first three chapters of [**H**], especially the first two. Note, however, that some conventions are different here.

This book will primarily use the language of schemes, rather than of varieties. The reader who prefers the more elementary approach of varieties, however, will often be able to mentally substitute the word *variety* for *scheme* without much loss, especially in the first few chapters.

With the exception of Appendix B, all schemes are assumed to be separated.

We often omit *Spec* when it is clear from the context; *e.g.*, $X(A)$ means $X(\operatorname{Spec} A)$ when A is a ring, $\mathbb{P}^n_A$ means $\mathbb{P}^n_{\operatorname{Spec} A}$, and $X \times_A B$ means $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ when A and B are rings.

The following definition gives slightly different names for some standard objects.

Definition 2.1. Let X be a scheme. Then a **vector sheaf** of rank r on X is a sheaf that is locally isomorphic to $\mathscr{O}_X^r$. A **line sheaf** is a vector sheaf of rank 1.

Note that a vector sheaf is what is often called a locally free sheaf, with the additional restriction that its rank be the same everywhere. A line sheaf is also called an invertible sheaf by many authors.

Varieties

Not all authors use the same definition of variety. Here we use:

Definition 2.2. Let k be a field. A **variety over** k, also called a k-variety, is an integral scheme, of finite type (and separated) over $\operatorname{Spec} k$. If it is also proper over k, then we say it is **complete**. A **curve** is a variety of dimension 1.

Note that, since k is not assumed to be algebraically closed, the set of closed points of a variety X is the set $X(\bar{k})$, modulo the action of $\operatorname{Aut}_k(\bar{k})$. Also, the residue field $K(P)$ for a closed point P will in general be a finite extension of k.

Note also that we have not assumed a variety to be geometrically integral. The advantage of this approach is that every irreducible closed subset of a variety will again be a variety, so that there is a natural one-to-one correspondence between the set of points of a variety and its set of subvarieties. This also agrees with the general philosophy that definitions should be weak.

Finally, note that X being a variety over k is not the same as X being defined over k (Definition 3.2). For example, let k be a number field. Then any variety over k can be transformed into a variety over $\mathbb{Q}$ merely by composing with the canonical morphism $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Q}$. On the other hand, not all varieties are defined over $\mathbb{Q}$; for example take the point in $\mathbb{A}^1_{\mathbb{Q}}$ corresponding to $\pm\sqrt{2}$.

More details on this situation appear in the next section.

§3. The field of definition of a variety

This section defines what it means for a variety to be "defined over" a field. We show that there is a well defined minimal field over which this is the case, and that it is true over all larger fields. Moreover, some information on the structure of this field is given.

We begin by describing what happens to a variety under base change to a larger field.

Proposition 3.1. *Let k be a field, let X be a variety over k, and let k' be a normal extension of k. Then every irreducible component of $X \times_k k'$ dominates X (under the projection $X \times_k k' \to X$), and all irreducible components are conjugate under the action of $\operatorname{Aut}_k(k')$.*

Proof. We may restrict to an open affine of finite type over k, so we may assume that X is a closed subvariety of $\mathbb{A}_k^n$ for some n. The first assertion is then obvious from the going-down theorem. The second assertion follows from ([**L 1**], Ch. I, Prop. 11), which states that if A is an integrally closed entire ring with field of fractions K, if L is a Galois extension of K, if B is the integral closure of A in L, and if $\mathfrak{p}$ is a maximal ideal in A, then all prime ideals of B lying over $\mathfrak{p}$ are conjugate under $\operatorname{Aut}_K(L)$. But by localizing, we may allow $\mathfrak{p}$ to be any prime ideal, and the restriction that L be Galois over K can be relaxed to L normal over K; the details of this are left to the reader. □

This motivates the following definition.

Definition 3.2. Let X be a variety over a field k, and let k' be an extension field of k. Then we say that X is **defined over** k' if all irreducible components of $(X \times_k k')_{\text{red}}$ are geometrically integral. If this is the case, then we also say that k' is a **field of definition** for X.

Remark 3.3. In particular, if $k' = k$, then this reduces to saying that X is defined over k if and only if it is geometrically integral.

Lemma 3.4. *Let $k \subseteq k'$ be fields, let X be a variety over k, and let $\{U_1, \dots, U_n\}$ be a cover of X by open subvarieties. Then X is defined over k' if and only if all U_i are defined over k'.*

Proof. Fix some i, and let U' be an irreducible component of $(U_i \times_k k')_{\text{red}}$. Then the closure of U' in $(X \times_k k')_{\text{red}}$ is an irreducible component of $(X \times_k k')_{\text{red}}$; since that irreducible component is geometrically integral, so is U'. Thus all U_i are defined over k'.

Now suppose that all U_i are defined over k', and let X' be an irreducible component of $(X \times_k k')_{\text{red}}$. Then X' is covered by irreducible components of $(U_i \times_k k')_{\text{red}}$; hence X' is geometrically integral. Thus the converse holds as well. □

Regular extensions

Definition 3.2 can be phrased in algebraic terms using the notion of regular field extensions.

Definition 3.5. A field extension K/k is **regular** if k is algebraically closed in K (*i.e.*, any element of K algebraic over k is already contained in k), and K is separable over k.

Lemma 3.6. *Let K/k be a field extension, and regard the algebraic closure $\bar{k}$ of k as a subfield of $\overline{K}$. Then the following conditions are equivalent.*

(i). *K is a regular extension of k;*
(ii). *K is linearly disjoint from $\bar{k}$ over k; and*
(iii). *the natural map $K \otimes_k \bar{k} \to K\bar{k}$ is injective.*

Proof. The equivalence (i) $\iff$ (ii) follows from ([**L 2**], Ch. VIII, Lemma 4.10). The equivalence of (ii) and (iii) is immediate from the definitions. □

Proposition 3.7. *A variety X over k is defined over k if and only if $K(X)$ is a regular extension of k.*

Proof. By Lemma 3.4, we may assume that X is affine.

Let A be the affine ring of X, and let $K = K(X)$, so that K is the field of fractions of A. Then $X \times_k \bar{k}$ is integral if and only if $A \otimes_k \bar{k}$ is entire.

First suppose that K is regular over k. Consider the composition of maps

$$A \times_k \bar{k} \hookrightarrow K \times_k \bar{k} \to K\bar{k}.$$

The first arrow is injective because $\bar{k}$ is flat over k. Lemma 3.6 implies that the second arrow is also injective, so $A \times_k \bar{k}$ is entire. Thus X is geometrically integral.

Conversely, assume that $A \otimes_k \bar{k}$ is entire. Then, for all finite subextensions L of $\bar{k}/k$, $A_L := A \otimes_k L$ is entire since it is a subring (by flatness). Let K' be its field of fractions. Let $\theta_1, \dots, \theta_d$ be a basis for L over k; then (by flatness) $\theta_1, \dots, \theta_d$, as elements of A_L, are linearly independent over A. Therefore they form a basis for K' over K, so $[K' : K] = d = [L : k]$. But it is easy to check that since A_L is integral over A, any map $A_L \to \overline{K}$ extending the injection $A \hookrightarrow K$ is injective. Thus $[KL : K] = [K' : K] = [L : k]$, so K and L are linearly disjoint over k. This holds for all L finite over k, so K and $\bar{k}$ are linearly disjoint over k. □

The minimal field of definition

We now show that a variety X over k has a unique minimal field of definition. This leads to a slightly different concept of the field of definition, in a different context; this is useful in its own right.

Definition 3.8. Let $k \subseteq k' \subseteq k_2$ be fields, let X be a scheme of finite type over k, and let Z be a closed subscheme of $X \times_k k_2$. Then we say that Z is **defined over** k' if there exists a subscheme Z' of $X \times_k k'$ such that $Z' \times_{k'} k_2 = Z$. If so, then we also say that k' is a **field of definition** for Z.

Lemma 3.9. *Let $k \subseteq k'$ be fields, and let X be a closed subvariety of $\mathbb{A}^n_k$. Then X is defined over k' (as a variety) if and only if every irreducible component of $(X \times_k \bar{k}')_{\mathrm{red}}$ is defined over k' (as a subscheme of $\mathbb{A}^n_{\bar{k}'}$).*

Proof. First, we immediately reduce to the case where $k = k'$.

If X is defined over k, then it is geometrically integral, and the only irreducible component of $(X \times_k \bar{k})_{\mathrm{red}}$ is $X \times_k \bar{k}$ itself, which is defined over k since it comes from X.

Conversely, suppose some irreducible component of $(X \times_k \bar{k})_{\mathrm{red}}$ is defined over k. Then there exists a scheme X' of $\mathbb{A}_k^n$ such that $X' \times_k \bar{k}$ is this irreducible component. But, by Proposition 3.1, $X' \times_k \bar{k}$ dominates X under the map $X \times_k \bar{k} \to X$, so since X' is integral, we must have $X' = X$. Thus $X \times_k \bar{k}$ is integral, so X is geometrically integral. □

Lemma 3.10. *Let $k \subseteq k_2$ be fields, let $n \in \mathbb{N}$, and let $\mathfrak{a}$ be an ideal in $k_2[X_1, \dots, X_n]$. Then there exists a field k_1 such that $k \subseteq k_1 \subseteq k_2$ and, for all fields k' with $k \subseteq k' \subseteq k_2$, $\mathfrak{a}$ is generated by elements of $k'[X_1, \dots, X_n]$ if and only if $k' \supseteq k_1$.*

Proof. By ([**C-L-O'S**], Ch. 2, § 7, Prop. 6), every ideal in a polynomial ring over a field has a unique reduced Gröbner basis. Let k_1 be the field generated over k by the coefficients of the elements of such a basis of $\mathfrak{a}$. Then the "if" part of the lemma is obvious.

Conversely, suppose $\mathfrak{a}$ is generated by elements of $k'[X_1, \dots, X_n]$ for some field k'. Let $\mathfrak{a}'$ be the ideal in $k'[X_1, \dots, X_n]$ generated by those elements. Then $\mathfrak{a}'$ has a reduced Gröbner basis with coefficients in k'. But the definition of reduced Gröbner basis involves only linear algebra in the coefficients of the polynomials, so the unique reduced Gröbner basis is preserved by enlarging the field of coefficients. In particular the reduced Gröbner bases of $\mathfrak{a}'$ and $\mathfrak{a}$ coincide; hence $k' \supseteq k_1$. □

Proposition 3.11. *Let $k \subseteq k_2$ be fields, let X be a scheme of finite type over k, and let Z be a closed subscheme of $X \times_k k_2$. Then there exists a unique minimal field of definition k_1 of Z: for all fields k' with $k \subseteq k' \subseteq k_2$, Z is defined over k' if and only if $k' \supseteq k_1$.*

Proof. In the special case where $X = \mathbb{A}_k^n$, this follows immediately by translating Lemma 3.10 into geometrical language. The general case follows by covering X by open affines U_i, which can then be regarded as closed subvarieties of $\mathbb{A}_k^{n_i}$. □

Proposition 3.12. *Let X be a variety over a field k, and let k_2 be a field extension of k containing an algebraic closure of k. Then there exists a unique minimal field of definition k_1 of X: for all fields k' with $k \subseteq k' \subseteq k_2$, X is defined over k' if and only if $k' \supseteq k_1$.*

Proof. By Lemma 3.4, we may reduce immediately to the case where X is affine. Thus we may consider X as a closed subvariety of $\mathbb{A}_k^n$.

If we replace k_2 with its algebraic closure, then a minimal field of definition for X exists by Lemma 3.9 and Proposition 3.11. But it is clear from Definition 3.2 that X is always defined over $\bar{k}$, so k_1 is contained in $\bar{k}$, which is contained in the original field k_2. □

The following proposition gives a good idea of the structure of the field k_1.

Proposition 3.13. *Let k be a field, let $\mathfrak{p}$ be a prime ideal in $\bar{k}[X_1,\dots,X_n]$, and let k' be the smallest field such that $\mathfrak{p}$ is generated by elements of $k'[X_1,\dots,X_n]$. Let L be the field of fractions of $\bar{k}[X_1,\dots,X_n]/\mathfrak{p}$, and let K be the subfield of L generated by k and the images of $X_1,\dots,X_n$. Then k' is a purely inseparable extension of the algebraic closure of k in K; moreover, K is separable over k if and only if k' is separable over k.*

Proof. See ([**W**], Ch. 1, Prop. 23). □

Corollary 3.14. *Let X be a variety over a field k, and let k' be the minimal field of definition of X. Then k' is a purely inseparable extension of the algebraic closure of k in $K(X)$; moreover, $K(X)$ is separable over k if and only if k' is separable over k.*

Proof. This follows by translating Proposition 3.13 into geometrical language. □

§4. Conventions and required knowledge in number theory

It is assumed that the reader has mastered the basics of algebraic number theory as presented, for example, in Part I of [**L 1**], especially the first five chapters.

In addition, the following definitions are used.

Number fields

A number field k has a canonical set of places, denoted M_k. This set is in one-to-one correspondence with the disjoint union of:

(i). the set of real embeddings $\sigma\colon k \hookrightarrow \mathbb{R}$;
(ii). the set of complex conjugate pairs $\{\sigma,\bar\sigma\}$ of embeddings $\sigma\colon k\hookrightarrow\mathbb{C}$; and
(iii). the set of nonzero prime ideals $\mathfrak{p}$ in the ring of integers R of k.

These places are referred to as **real places**, **complex places**, and **non-archimedean places**, respectively. In addition, an **archimedean place** is a real or complex place.

These places have **almost-absolute values** $\|\cdot\|_v$ defined by

$$\|x\|_v = \begin{cases} |\sigma(x)|, & \text{if } v \text{ is real, corresponding to } \sigma\colon k\hookrightarrow\mathbb{R};\\ |\sigma(x)|^2, & \text{if } v \text{ is complex, corresponding to } \sigma,\bar\sigma\colon k\hookrightarrow\mathbb{C};\\ (R:\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}}(x)}, & \text{if } v \text{ is non-archimedean, corresponding to } \mathfrak{p}\subseteq R\end{cases}$$

(In the last of the above three cases, $\operatorname{ord}_{\mathfrak{p}}(x)$ is the **order** of x at $\mathfrak{p}$; *i.e.*, the exponent of $\mathfrak{p}$ in the factorization of the fractional ideal (x). This requires $x\ne 0$; we also define $\|0\|_v=0$.) These are not necessarily genuine absolute values, since the almost absolute value associated to a complex place does not satisfy the triangle inequality.

Instead of the triangle inequality, however, we have that if $a_1,\dots,a_n\in k$, then

$$\left\|\sum_{i=1}^n a_i\right\|_v \le n^{N_v}\max_{1\le i\le n}\|a_i\|_v, \tag{4.1}$$

where

$$N_v = \begin{cases} 1 & \text{if } v \text{ is real;} \\ 2 & \text{if } v \text{ is complex; and} \\ 0 & \text{if } v \text{ is non-archimedean.} \end{cases} \tag{4.2}$$

In particular, if v is non-archimedean, then $\|\cdot\|_v$ obeys something stronger than the triangle inequality:

$$\|x+y\|_v \le \max(\|x\|_v, \|y\|_v). \tag{4.3}$$

In addition, note that we have

$$\sum_{v \in M_k} N_v = [k : \mathbb{Q}] \tag{4.4}$$

for any number field k.

In addition, if L is a finite extension of k, $v \in M_k$, and $x \in k$, then

$$\prod_{\substack{w \in M_L \\ w \mid v}} \|x\|_w = \|x\|_v^{[L:k]}. \tag{4.5}$$

Furthermore, these places satisfy a **product formula**:

$$\prod_{v \in M_k} \|x\|_v = 1 \qquad \text{for all } x \in k^*. \tag{4.6}$$

The set of archimedean places of a number field k is denoted $S_\infty(k)$, or just S_∞ if it is clear what k is.

Function fields

Recall that a function field is the field $k := K(X)$ of rational functions on some nonsingular projective curve X over a field k_0. Fix a constant $c > 1$. Then define a set M_k of places of k by defining, for each closed point $P \in X$, a place v with absolute value defined by

$$\|x\|_v = c^{-[K(P):k_0]\operatorname{ord}_P x}, \tag{4.7}$$

where ord_P is the order of vainishing of the rational function x at P. (Again, (4.7) requires that x be nonzero; we of course define $\|0\|_v = 0$ for all $v \in M_k$.) Then these absolute values again satisfy the product formula (4.6), as well as the formula (4.5) for extensions of fields. It is customary to take $c = e$, so that $\log \|x\|_v$ will take on integral values. Or, with finite fields, one could take $c = \#k_0$, to provide some parallels with the number field case.

In the number field case, there is really only one choice for M_k, and the normalizations of $\|\cdot\|_v$ do not depend on additional choices. This is not the case in the function field case, however. Not only do the normalizations of $\|\cdot\|_v$ depend on c and on k_0, but also the set M_k may vary for a given function field k. For example, with the field $k := \mathbb{C}(X,Y)$, we may take $\mathbb{C}(X)$ as the field of constants and treat Y as an indeterminate, or vice versa. Therefore, we will assume that, whenever a function field k is given, its set M_k of places, and the normalizations of its absolute values, are given with it. Furthermore, an extension L/k of function fields is assumed to be one for which the set M_L is the set of all places extending places in M_k, and c and k_0 coincide. Then, in the notation of the preceding paragraph, $L = K(X')$ for some curve X' provided with a finite morphism to X.

For a function field k, all places $v \in M_k$ are non-archimedean. Therefore, by (4.3), the set

$$\{x \in k \mid \|x\|_v \le 1 \text{ for all } v \in M_k\}$$

is a field, called the **field of constants** of k. If X is defined over k_0, then k_0 coincides with this field of constants, by Proposition 3.7 and Lemma 3.6. Moreover, this remains true if k_0 is enlarged. Therefore it will often be assumed that X is defined over k_0.

For a function field k, let $N_v = 0$ for all $v \in M_k$; then (4.1) and (4.2) hold also in the function field case. In addition:

Convention 4.8. If k is a function field, then we adopt the convention that $[k:\mathbb{Q}] = 0$.

With this convention, (4.4) holds for function fields as well.

If k is a function field, then we let $S_\infty(k) = \emptyset$.

Global fields

Definition 4.9. Recall that a **global field** is either a number field or a function field.

If k is a global field, we let $\overline{M}_k$ denote the disjoint union of M_L for all finite extensions L of k.

Local fields

Definition 4.10. A **local field** is the completion of a global field at one of its places.

The set of local fields (as defined here) coincides with the disjoint union of:

(i). The set of finite extensions of $\mathbb{Q}_p$ for some rational prime p,
(ii). The set of finite extensions of $k_0((T))$ for some field k_0, and
(iii). $\mathbb{R}$ and $\mathbb{C}$.

If k is a global field and $v \in M_k$, then the almost-absolute value $\|\cdot\|_v$ extends uniquely to an almost-absolute value on the completion k_v. In that case, we may drop the subscript v, since the place is implicit from the fact that we are dealing with elements of k_v: $\|x\|$ for $x \in k_v$.

For a local field k_v as above, we let $M_{k_v} = \{v\}$. Of course, there is no product formula in this case.

§5. Associated points, rational maps, and rational sections

When dealing with schemes that are not necessarily reduced, we use a definition of rational map and rational section that is a bit different from the usual definition; see Definition 5.3. This definition is based on schematic denseness.

Definition 5.1. Let U be an open subset of a scheme X. We say that U is **schematically dense** in X if its schematic closure (the schematic image of the open immersion $U \hookrightarrow X$, or the smallest closed subscheme of X containing U) is all of X.

Often it is convenient to think of schematic denseness in terms of the associated points of X.

Proposition 5.2. *An an open subset U of a noetherian scheme X is schematically dense if and only if it contains all associated points of X.*

Proof. It suffices to show this when X is an affine scheme. Let A be the affine ring of X, let $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ be a minimal primary decomposition of the ideal (0) in A, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for each i. Then $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the primes corresponding to the associated points of X.

First suppose that U contains all associated points of X. Let $\mathfrak{a}$ be the ideal corresponding to the schematic closure of U, and let $f \in \mathfrak{a}$. For all i, $\operatorname{Spec} A/\mathfrak{a}$ contains an open neighborhood of $\mathfrak{p}_i$; therefore $(A/\mathfrak{a})_{\mathfrak{p}_i/\mathfrak{a}} = A_{\mathfrak{p}}$. In particular, $f = 0$ in $A_{\mathfrak{p}}$, so $\operatorname{Ann}(f)$ meets $A \setminus \mathfrak{p}$. Thus, for all i there exists $x_i \in \operatorname{Ann}(f)$ such that $x_i \notin \mathfrak{p}_i$. It follows by an easy exercise that there exists $x \in \operatorname{Ann}(f)$ such that $x \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$. But then x is not a zero divisor, by ([**L 2**], X Prop. 2.9). This can happen only if $f = 0$, so $\mathfrak{a} = 0$ and thus U is schematically dense in X.

Conversely, suppose that U does not contain the point corresponding to $\mathfrak{p}_i$. After renumbering the indices, we may assume that $i = n$ and that $\{i \mid \mathfrak{p}_i \supseteq \mathfrak{p}_n\} = \{1, \dots, r\}$ for some $r < n$. Let

$$\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r.$$

Since the chosen primary decomposition was minimal, we have $\mathfrak{a} \neq (0)$. We claim that $U \subseteq \operatorname{Spec} A/\mathfrak{a}$. Indeed, let $\mathfrak{p}$ be a prime ideal corresponding to a point in U. Since U is open and since $\mathfrak{p}_n \notin U$, we have $\mathfrak{p} \not\supseteq \mathfrak{p}_n$, and therefore $\mathfrak{p} \not\supseteq \mathfrak{p}_i$ for all $i > r$. Let $S = A \setminus \mathfrak{p}$. By ([**A-M**], Prop. 4.8(i)) and ([**A-M**], Prop. 4.9), we then have

$$(0) = \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i = \bigcap_{i=1}^{r} S^{-1}\mathfrak{q}_i = S^{-1}\mathfrak{a}.$$

Thus $\mathfrak{p} \in \operatorname{Spec} A/\mathfrak{a}$ (since $S^{-1}\mathfrak{a} = (0)$ implies $\mathfrak{a} \subseteq \mathfrak{p}$), and $(A/\mathfrak{a})_{\mathfrak{p}} = A_{\mathfrak{p}}$. This holds for all $\mathfrak{p} \in U$, so U is an open subscheme of $\operatorname{Spec} A/\mathfrak{a}$. This shows that U is not schematically dense in X. □

Definition 5.3. Let X be a scheme.

(a). If X is noetherian, then a **rational map** $f\colon X \dashrightarrow Y$ from X to another scheme Y is an equivalence class of pairs (U, f), where U is a schematically dense open subset of X, $f\colon U \to Y$ is a morphism, and (U, f) is said to be equivalent to (U', f') if f and f' agree on $U \cap U'$.

(b). If X is noetherian, then a **rational function** on X is a class of pairs (U, f), where U is a schematically dense open subset of X, f is a regular function on U, and the equivalence relation is similar to the one in part (a).

(c). A **rational section** of a sheaf $\mathscr{F}$ on X is a section of $\mathscr{F}$ over a schematically dense open set.

(d). A rational section of a line sheaf $\mathscr{L}$ on X is said to be **invertible** if it has an inverse over a schematically dense open subset.

Remark. The relations in parts (a) and (b) above are equivalence relations because Y is separated, and because the open sets in question are schematically dense.

§6. Cartier divisors and associated line sheaves

The above definition of invertible rational section of a line sheaf meshes well with the notion of a Cartier divisor.

We begin by recalling the definition of a Cartier divisor.

Definition 6.1. Let X be a scheme.

(a). The sheaf $\mathscr{K}$, called the **sheaf of total quotient rings** of $\mathscr{O}_X$, is the sheaf associated to the presheaf $U \mapsto S^{-1}\mathscr{O}_X(U)$, where S is the multiplicative system of elements which are not zero divisors.

(b). Let $\mathscr{O}_X^*$ denote the sheaf of invertible elements of $\mathscr{O}_X$. Then a **Cartier divisor** on X is a global section of the sheaf $\mathscr{K}^*/\mathscr{O}_X^*$.

The set of Cartier divisors forms a group, which is written additively instead of multiplicatively by analogy with Weil divisors.

By standard properties of sheaves, a Cartier divisor can be described by giving an open cover $\{U_i\}$ of X and sections $f_i \in \mathscr{K}(U_i)$ such that f_i/f_j lies in $\mathscr{O}_X(U_i \cap U_j)^*$ for all i and j. Such a description is called a **system of representatives** for the Cartier divisor. If $\{(U_i, f_i)\}_{i\in I}$ is a system of representatives for a Cartier divisor D, then $D\big|_{U_i} = (f_i)$ for all i. If the scheme X is integral, which is true in most applications, then $\mathscr{K}$ is just the constant sheaf $K(X)$, and therefore in a system of representatives all f_i may be taken to lie in $K(X)$.

A Cartier divisor is **principal** if it lies in the image of the map

$$\Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathscr{O}_X^*).$$

If X is integral, then this is equivalent to the assertion that all f_i can be taken equal to the same $f \in K(X)$. Two Cartier divisors are **linearly equivalent** if their difference is principal. A Cartier divisor is **effective** if it has a system of representatives $\{(U_i, f_i)\}_{i\in I}$ such that f_i lies in $\mathscr{O}_X(U_i)$ for all i. If D is a Cartier divisor, its **support**, denoted $\operatorname{Supp} D$, is the set

$$\operatorname{Supp} D = \{P \in X \mid D_P \notin \mathscr{O}_{X,P}^*\}$$

(where $D_P \in \mathscr{K}_P$ denotes the germ of D at P). The support of D is a Zariski-closed subset of X. More concretely, if $\{(U_i, f_i)\}_{i\in I}$ is a system of representatives for D, then $(\operatorname{Supp} D) \cap U_i$ equals the set of all $P \in U_i$ such that f_i is not an invertible element of the local ring $\mathscr{O}_{X,P}$ at P. If D is a Cartier divisor with $\operatorname{Supp} D = \emptyset$, then $D = (1)$.

For more information on Cartier divisors and how they compare to Weil divisors, see ([**H**], II §6).

Lemma 6.2. *Let A be a commutative noetherian ring. Let $X = \operatorname{Spec} A$, and let U be an open subset of X containing all associated points. Let S be the multiplicative system of (nonzero) elements of A which are not zero divisors. Then $\mathscr{O}_X(U) \subseteq S^{-1}A$. Conversely, for any $s \in S^{-1}A$, there exists an open subset U, containing all associated points of X, such that $s \in \mathscr{O}_X(U)$.*

Proof. Let s be a section in $\mathscr{O}_X(U)$. Since the complement of U is a closed subset not containing any associated point of X, there exists $f \in A$ which vanishes on the complement of U, yet is not contained in any associated prime of A. The former condition implies that $s \in A_f$, since $D(f) \subseteq U$; the latter implies that f is not a zero divisor, by ([**L 2**], X Prop. 2.9). Hence $f \in S$ and $s \in A_f \subseteq S^{-1}A$.

Conversely, suppose $s \in S^{-1}A$. Then $s \in A_f$ for some $f \in S$. In particular, by ([**L 2**], X Prop. 2.9), f is not contained in any associated prime of A. Thus $s \in \mathscr{O}_X(U)$ for some open U containing all the associated points of X. □

Definition 6.3. Let X be a noetherian scheme, let $\mathscr{L}$ be a line sheaf on X, and let s be an invertible rational section of $\mathscr{L}$. Then, for any open affine subset $U = \operatorname{Spec} A$ of X over which $\mathscr{L}$ is trivial, the restriction of s to U defines an element of $S^{-1}A$, where S is as in Lemma 6.2. Since s is invertible, this element actually lies in $(S^{-1}A)^*$. This element depends on the choice of trivialization: changing the trivialization multiplies the element by an element of A^*. Therefore, this gives a well defined section of $\mathscr{K}^*/\mathscr{O}^*$ over U. These sections glue to give a global section of $\mathscr{K}^*/\mathscr{O}^*$ over X. This section, regarded as a Cartier divisor, is called the **associated Cartier divisor**, and is denoted (s).

Proposition 6.4. *Let X be a noetherian scheme.*

(a). *Let f be a rational function on X. Then f may be regarded as an invertible rational section of the trivial line sheaf $\mathscr{O}_X$, and the definition of (f) as such coincides with the principal divisor (f).*

(b). *Let $\mathscr{L}$ be a line sheaf on X and let s be an invertible rational section of $\mathscr{L}$. Then s is a global (regular) section of $\mathscr{L}$ if and only if (s) is effective.*

(c). *Let $\phi\colon X' \to X$ be a morphism of noetherian schemes, let $\mathscr{L}$ be a line sheaf on X, and let s be an invertible rational section of $\mathscr{L}$ which is defined and nonzero at the images (under ϕ) of all associated points of X'. Then $\phi^* s$ is an invertible rational section of $\phi^*\mathscr{L}$, and $(\phi^* s) = \phi^*(s)$.*

(d). *Let s_1 and s_2 be invertible rational sections of line sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Then $s_1 \otimes s_2$ is an invertible rational section of $\mathscr{L}_1 \otimes \mathscr{L}_2$, and $(s_1 \otimes s_2) = (s_1) + (s_2)$.*

Proof. These all follow immediately from the definition. □

Recall from ([**H**], II §6) the definition of the sheaf $\mathscr{O}(D)$ associated to a Cartier divisor D on a scheme X (Hartshorne denotes it $\mathscr{L}(D)$). It is a subsheaf of $\mathscr{K}$.

Definition 6.5. Let D be a Cartier divisor on a noetherian scheme X . The global section $1 \in \mathscr{K}$ defines a rational section, called the **canonical section** of $\mathscr{O}(D)$.

In addition to ([**H**], II Prop. 6.13), we have the following properties.

Proposition 6.6. *Let D be a Cartier divisor on a noetherian scheme X .*

(a). *If s is the canonical section of $\mathscr{O}(D)$, then $(s) = D$.*

(b). *If $\phi\colon X' \to X$ is a morphism of noetherian schemes such that none of the associated points of X' are taken into the support of D, then $\mathscr{O}(\phi^* D) \cong \phi^* \mathscr{O}(D)$.*

Proof. This is left as an exercise for the reader. □

§7. Big divisors and Kodaira's lemma

——— Not written yet.

§8. Descent

——— Not written yet. Should it go earlier? See Serre, Groupes Alg. et Corps de Classes, p. 108 (Ch. V No. 20).

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