A MORE GENERAL ABC CONJECTURE

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In this note we formulate a conjecture generalizing both the *abc* conjecture of Masser-Oesterlé and the author's diophantine conjecture for algebraic points of bounded degree. We also show that the latter conjecture implies the new conjecture.

As with most of the author's conjectures, this new conjecture stems from analogies with Nevanlinna theory. In this particular case the conjecture corresponds to relacing the usual counting function of Nevanlinna theory with a truncated counting function. In particular, the *abc* conjecture of Masser and Oesterlé corresponds to Nevanlinna's Second Main Theorem with truncated counting functions applied to the divisor $[0]+[1]+[\infty]$ on $\mathbb{P}^1$.

The first section of this paper introduces the notation that will be used throughout the paper. Section 2 formulates the new conjecture and discusses some examples related to the new conjecture, including an "*abcde*... conjecture" and a conjecture of Buium. The third and final section of this paper shows that the new conjecture is implied by the (apparently weaker) older conjecture without truncated counting functions.

§1. Notation

This section briefly recalls the Nevanlinna-based notation from ([**V**], §3.2) that will be needed for stating the conjecture.

Let k be a global field. Its set of places will be denoted M_k. Each place $v \in M_k$ has an associated almost absolute value $\|\cdot\|_v$, normalized as follows. If k is a number field, then let $\mathscr{O}_k$ denote its ring of integers. A non-archimedean place $v \in M_k$ corresponds to a nonzero prime ideal $\mathfrak{p} \subseteq \mathscr{O}_k$, and we set $\|x\|_v = (\mathscr{O}_k : \mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}} x}$ if $x \in k^\times$ (and $\|0\|_v = 0$). If v is archimedean, then v corresponds to a real embedding $\sigma\colon k \hookrightarrow \mathbb{R}$ or a complex conjugate pair of complex embeddings $\sigma, \bar\sigma\colon k \hookrightarrow \mathbb{C}$, and we set $\|x\|_v = |\sigma(x)|$ or $\|x\|_v = |\sigma(x)|^2$, respectively. (In the latter case the triangle inequality fails to hold, hence the terminology "almost absolute value.") If k is a function field, then $k = K(C)$ for some smooth projective curve $C = C_k$ over the field of

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constants $k_0 \subseteq k$; places $v \in M_k$ correspond bijectively to closed points $p \in C_k$, and we set $\|x\|_v = \exp(-[K(p) : k_0]\operatorname{ord}_p(x))$ if $x \in k^\times$.

If k is a global field and $v \in M_k$, then let k_v denote the completion of k at v, and let $\mathbb{C}_v$ denote the completion of the algebraic closure of k_v.

The proofs in this paper will use Arakelov theory, some of whose language is as follows. The general idea is that the number field case should mimic as closely as possible the situation encountered in the function field case, where one has the projective curve C_k, and one can work with intersection theory on a proper scheme over C_k. In the number field case this is accomplished by formally adding analytic information for each archimedean place; hence the role of C_k is played by an arithmetic scheme $\mathbb{M}_k$ consisting of $\operatorname{Spec}\mathscr{O}_k$, with finitely many points added, corresponding to the archimedean places. Therefore, one can think of $\mathbb{M}_k$ as an object whose closed points are in canonical bijection with M_k. We also define the **non-archimedean part** $(\mathbb{M}_k)_{\mathrm{na}}$ of $\mathbb{M}_k$ to be $\operatorname{Spec}\mathscr{O}_k$. In the function field case, we set $\mathbb{M}_k = (\mathbb{M}_k)_{\mathrm{na}} = C_k$.

In this paper, a **variety** over a field k is an integral separated scheme of finite type over k. If k is a global field (which we assume from now on), then an **arithmetic variety** $\mathscr{X}$ over $\mathbb{M}_k$ is an integral scheme $\mathscr{X}_{\mathrm{na}}$, flat, separated, and of finite type over $(\mathbb{M}_k)_{\mathrm{na}}$, together with some analytic information at the archimedean places (which will play no role in this paper). An arithmetic variety $\mathscr{X}$ over $\mathbb{M}_k$ is **proper** over $\mathbb{M}_k$ if $\mathscr{X}_{\mathrm{na}}$ is proper over $(\mathbb{M}_k)_{\mathrm{na}}$.

If X is a variety over k, then a **model** for X is an arithmetic variety $\mathscr{X}$ over $\mathbb{M}_k$, together with an isomorphism $X \cong \mathscr{X} \times_{\mathbb{M}_k} k := \mathscr{X}_{\mathrm{na}} \times_{(\mathbb{M}_k)_{\mathrm{na}}} k$.

Let X be a complete variety over a global field k, and let $\mathscr{X}$ be a proper model over $\mathbb{M}_k$. An algebraic point $P \in X(\bar{k})$ determines a map $\sigma\colon \mathbb{M}_E \to \mathscr{X}$ over $\mathbb{M}_k$ (that is, a map $\sigma_{\mathrm{na}}\colon (\mathbb{M}_E)_{\mathrm{na}} \to \mathscr{X}_{\mathrm{na}}$ over $(\mathbb{M}_k)_{\mathrm{na}}$), where $E = k(P)$.

A **Cartier divisor** D on an arithmetic variety $\mathscr{X}$ over $\mathbb{M}_k$ is a Cartier divisor D_{na} on $\mathscr{X}_{\mathrm{na}}$, together with Green functions $g_{D,v}$ for D on $\mathscr{X}(\mathbb{C}_v) := \mathscr{X}_{\mathrm{na}}(\mathbb{C}_v)$ for all archimedean $v \in M_k$. The Green functions are taken to be normalized corresponding to $-\log\|\cdot\|_v$. A **principal** Cartier divisor may be defined in the obvious way, using $g_{(f),v} = -\log\|f\|_v$, and one obtains the notion of linear equivalence of Cartier divisors on $\mathscr{X}$. A **line sheaf** $\mathscr{L}$ on $\mathscr{X}$ is a line sheaf $\mathscr{L}_{\mathrm{na}}$ on $\mathscr{X}_{\mathrm{na}}$, together with metrics at the archimedean places. We have a natural bijection between the group of divisor classes and the group of isomorphism classes of line sheaves on $\mathscr{X}$.

One may regard $\mathbb{M}_k$ as an arithmetic variety over itself. Let D be a Cartier divisor on $\mathbb{M}_k$, and let $v \in M_k$. Then we define the **degree** of D at v as follows. If v is archimedean, then $\mathbb{M}_k(\mathbb{C}_v)$ consists of just one point, so $g_{D,v}$ is just a real number, and we let $\deg_v D = g_{D,v}$. Otherwise, v corresponds to a closed point on $(\mathbb{M}_k)_{\mathrm{na}}$, also denoted v. Let n_v be the multiplicity of D_{na} at that point, let $K(v)$ denote the residue field at v, and let

$$\mu(K(v)) = \begin{cases} \log \#K(v) & \text{if } k \text{ is a number field; or} \\ [K(v) : k_0] & \text{if } k \text{ is a function field with field of constants } k_0. \end{cases}$$

We then define

$$\deg_v D = n_v \mu(K(v)) .$$

For subsets $S \subseteq M_k$, let

$$\deg_S D = \sum_{v \in S} \deg_v D .$$

For $S = M_k$, we define $\deg D = \deg_{M_k} D$. If $f \in k^\times$, then the Artin-Whaples product formula implies that $\deg(f) = 0$. Thus the degree $\deg \mathscr{L}$ of a line sheaf $\mathscr{L}$ on $\mathbb{M}_k$ is well defined, via the corresponding divisor class.

If E is a finite extension of k, then $\mathbb{M}_E$ may be regarded as an arithmetic variety over $\mathbb{M}_k$, as well as an arithmetic variety over itself. The definitions of Cartier divisors and line sheaves on $\mathbb{M}_E$ do not coincide in this case (due to possible archimedean places), but there are obvious translations back and forth.

Let k be a global field, let X be a complete variety over k, let $\mathscr{X}$ be a proper model for X over $\mathbb{M}_k$, let $\mathscr{L}$ be a line sheaf on $\mathscr{X}$, let $P \in X(\bar{k})$, let E be a finite extension of k containing $k(P)$, and let $\sigma\colon \mathbb{M}_E \to \mathscr{X}$ correspond to P. Then the height of P (relative to $\mathscr{L}$ and k) satisfies

$$h_{\mathscr{L},k}(P) = \frac{1}{[E:k]} \deg \sigma^* \mathscr{L} .$$

It is linear and functorial in $\mathscr{L}$, and is independent of the choice of E. If D is a Cartier divisor on $\mathscr{X}$ such that $P \notin \operatorname{Supp} D$, then $\sigma^* D$ is defined, and we have

$$h_{\mathscr{O}(D),k}(P) = \frac{1}{[E:k]} \deg \sigma^* D .$$

Let S be a finite set of places of $\mathbb{M}_k$ containing the archimedean places, and let $T = \{w \in M_E \bigm| w \mid v \text{ for some } v \in S\}$. Then we may split $h_{\mathscr{O}(D),k}(P)$ into two terms:

$$h_{\mathscr{O}(D),k}(P) = m_{k,S}(D,P) + N_{k,S}(D,P) ,$$

where

$$m_{k,S}(D,P) := \frac{1}{[E:k]} \deg_T \sigma^* D \qquad \text{and} \qquad N_{k,S}(D,P) := \frac{1}{[E:k]} \deg_{M_E \setminus T} \sigma^* D$$

are called the **proximity function** and **counting function**, respectively. (The names and notation come from Nevanlinna theory.)

We also define the notation $w \mid S$ to mean $w \in T$, where T is defined above.

If $\mathscr{X}'$ is another model for $\mathscr{X}$, and if D' is a Cartier divisor on $\mathscr{X}'$ coinciding on X with D, then we have

$$\begin{aligned} h_{\mathscr{O}(D'),k}(P) &= h_{\mathscr{O}(D),k}(P) + O(1) , \\ m_{k,S}(D',P) &= m_{k,S}(D,P) + O(1) , \qquad \text{and} \\ N_{k,S}(D',P) &= N_{k,S}(D,P) + O(1) \end{aligned}$$

for all $P \in X(\bar{k})$. Therefore the height, proximity, and counting functions may be discussed in terms of Cartier divisors on X if their values are only needed up to a bounded function.

For more details on height, proximity, and counting functions, including an alternative definition using Weil functions, see ([**V**], §3.4).

Since k and S will often be fixed for a given discussion, they will often be omitted from the notation.

All places in $M_k \setminus S$ are non-archimedean; hence the counting function may be written

$$N_{k,S}(D,P) = \frac{1}{[E:k]} \sum_{\substack{w \in M_E \\ w \nmid S}} n_v \cdot \mu(K(v)) \;.$$

Here n_v is the multiplicity of v in the divisor $\sigma^* D$, as above, and $\mu(K(v))$ is as defined earlier.

We may then define the **truncated counting function**

$$N^{(1)}_{k,S}(D,P) = \frac{1}{[k(P):k]} \sum_{\substack{w \in M_{k(P)} \\ w \nmid S}} \min\{1, n_v\} \cdot \mu(K(v))$$

for effective Cartier divisors D on $\mathscr{X}$ (*i.e.*, Cartier divisors such that D_{na} is effective) and for points $P \in X(\bar{k}) \setminus \operatorname{Supp} D$. The truncated counting function is not necessarily additive or functorial in D.

We next define some quantities related to the logarithm of the discriminant of a number field, or the genus of the curve corresponding to a function field. Let E be a finite extension of a global field k of characteristic 0. Then we have a finite morphism $\mathbb{M}_E \to \mathbb{M}_k$. In this situation, let $R_{E/k}$ denote the Cartier divisor on $\mathbb{M}_E$ such that $(R_{E/k})_{\mathrm{na}}$ is the ramification divisor of the corresponding map $(\mathbb{M}_E)_{\mathrm{na}} \to (\mathbb{M}_k)_{\mathrm{na}}$, and such that the corresponding Green functions are all zero. We then define

$$d_k(E) = \frac{1}{[E:k]} \deg R_{E/k} \qquad \text{and} \qquad d_{k,S}(E) = \frac{1}{[E:k]} \deg_{M_E \setminus S} R_{E/k} \;,$$

where S is a finite subset of M_k containing the archimedean places, and $M_E \setminus S$ means $\{w \in M_E \mid w \nmid S\}$. We note that:

(i). If k is a number field and D_k denotes its discriminant, then

$$d_k(E) = \frac{1}{[E:k]} \log|D_E| - \log|D_k| \;;$$

(ii).

$$0 \le d_k(E) - d_{k,S}(E) \le O_{[E:k],S}(1) \;; \tag{1.1}$$

and

(iii). if F is a finite extension of E, then

$$d_{k,S}(F) - d_{k,S}(E) = \frac{1}{[F:k]} \deg_{M_F \setminus S} R_{F/E} . \tag{1.2}$$

For $P \in X(\bar{k})$, we define $d_k(P) = d_k(k(P))$ and $d_{k,S}(P) = d_{k,S}(k(P))$.

Let X be a smooth variety. A **normal crossings divisor** on X is a divisor that, for all points $P \in X$, can be represented in the completed local ring $\widehat{\mathscr{O}}_{P,X}$ by a principal divisor $(x_1 \cdots x_r)$, where $x_1, \dots, x_r$ form a part of a regular sequence for $\widehat{\mathscr{O}}_{P,X}$. (A normal crossings divisor must therefore be effective, and all irreducible components of its support must occur with multiplicity 1.) We also say that a divisor D **has normal crossings** if it is a normal crossings divisor.

Finally, a divisor D or line sheaf $\mathscr{L}$ on a complete variety X is said to be **big** if there is a constant $c > 0$ such that

$$h^0(X, \mathscr{O}(nD)) \ge cn^{\dim X} \qquad \text{or} \qquad h^0(X, \mathscr{L}^{\otimes n}) \ge cn^{\dim X} ,$$

respectively, for all sufficiently large and divisible integers n. By Kodaira's lemma (see, for example, ([**V**], Prop. 1.2.7)), if D is a big divisor and A is an ample divisor, then $nD - A$ is linearly equivalent to an effective divisor for sufficiently large and divisible integers n.

§2. The conjecture

We begin by recalling from ([**V**], Conj. 5.2.6) the general conjecture on algebraic points of bounded degree.

Throughout this section, k is a global field of characteristic 0 and S is a finite set of places of k containing the archimedean places.

Conjecture 2.1. *Let X be a smooth complete variety over k, let D be a normal crossings divisor on X, let $\mathscr{K}$ denote the canonical line sheaf on X, let $\mathscr{A}$ be a big line sheaf on X, let $r \in \mathbb{Z}_{>0}$, and let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z = Z(k, S, X, D, \mathscr{A}, r, \epsilon) \subsetneq X$ such that*

$$h_{\mathscr{K}}(P) + m(D,P) \le d_k(P) + \epsilon\, h_{\mathscr{A}}(P) + O(1) \tag{2.1.1}$$

for all $P \in X(\bar{k}) \setminus Z$ with $[k(P):k] \le r$.

(In [**V**], the term $d_k(P)$ had a factor $\dim X$ in front, but in recent years it has become apparent that the inequality may be true without this factor.)

Since $h_{\mathscr{O}(D)}(P) = m(D,P) + N(D,P) + O(1)$, (2.1.1) is equivalent to

$$N(D,P) + d_k(P) \ge h_{\mathscr{K}(D)}(P) - \epsilon\, h_{\mathscr{A}}(P) - O(1) . \tag{2.2}$$

One may then ask whether $N(D,P)$ could be replaced by the truncated counting function:

Conjecture 2.3. *Conjecture 2.1 holds with (2.1.1) replaced by*

$$N^{(1)}(D,P) + d_k(P) \ge h_{\mathscr{K}(D)}(P) - \epsilon\, h_{\mathscr{A}}(P) - O(1)\ . \tag{2.3.1}$$

We always have $N^{(1)}(D,P) \le N(D,P)$, so Conjecture 2.3 is obviously stronger than Conjecture 2.1. The main goal of the next section will be to show that the converse holds; *i.e.*, Conjecture 2.1 implies Conjecture 2.3.

First, however, we shall show how Conjecture 2.3 generalizes the "*abc* conjecture" of Masser and Oesterlé, which is the following:

Conjecture 2.4 (Masser-Oesterlé). *For all* $\epsilon > 0$ *there is a constant* $C > 0$ *such that for all* $a, b, c \in \mathbb{Z}$ *with* $a + b + c = 0$ *and* $(a,b,c) = 1$, *we have*

$$\max\{|a|,|b|,|c|\} \le C \cdot \prod_{p|abc} p^{1+\epsilon}\ . \tag{2.4.1}$$

To relate this conjecture to Conjecture 2.3, we begin by translating (2.4.1) into the language of Section 1. The triple (a,b,c) determines a point $P := [a : b : c]$ on $\mathbb{P}^2$, which in fact lies on the line $x_0 + x_1 + x_2 = 0$. The height of this point is $h_{\mathscr{O}(1),\mathbb{Q}}(P) = h(P) = \log\max\{|a|,|b|,|c|\}$ (since $(a,b,c) = 1$). The relative primeness condition also implies that the curve on $\mathbb{P}^2_{\mathbb{Z}}$ corresponding to P meets the divisor $[x_0 = 0]$ at a prime p if and only if $p \mid a$. Similarly, it meets the divisors $[x_1 = 0]$ and $[x_2 = 0]$ at p if and only if $p \mid b$ and $p \mid c$, respectively. Let D be the divisor $[x_0 = 0] + [x_1 = 0] + [x_2 = 0]$ on $\mathbb{P}^2$. Then

$$N^{(1)}_{\mathbb{Q},\{\infty\}}(D,P) = \sum_{p|abc} \log p\ .$$

Thus (2.4.1) can be written

$$h(P) \le (1+\epsilon) N^{(1)}(D,P) + O_\epsilon(1)$$

or (with a different ϵ)

$$(1-\epsilon)h(P) \le N^{(1)}(D,P) + O_\epsilon(1) \tag{2.5}$$

for all $P \in \mathbb{P}^2(\mathbb{Q})$ lying on the line $x_0 + x_1 + x_2 = 0$. Let X be this line. Its canonical line sheaf is $\mathscr{K} \cong \mathscr{O}(-2)$, and $D\big|_X$ consists of three distinct points, so $\mathscr{K}(D) \cong \mathscr{O}(1)$; hence $h(P) = h_{\mathscr{K}(D)}(P) + O(1)$. Also let $\mathscr{A} = \mathscr{O}(1)$; then (2.5) becomes

$$N^{(1)}(D,P) \ge h_{\mathscr{K}(D)}(P) - \epsilon\, h_{\mathscr{A}}(P) - O_\epsilon(1)\ ,$$

which coincides with (2.3.1) since we are dealing with rational points and therefore $d(P) = 0$ for all P.

Thus, it follows that the *abc* conjecture coincides with the special case of Conjecture 2.3 when $k=\mathbb{Q}$, $S=\{\infty\}$, $X=\mathbb{P}^1$, D consists of three distinct points, and $r=1$. Conjecture 2.3 can therefore be viewed as doing for the *abc* conjecture what Conjecture 2.1 did for Roth's theorem.

One may wonder what this says about what the exponent should be for the *abc* conjecture in more than three variables (e.g., $a+b+c+d=0$). Conjecture 2.3 suggests that the exponent should still be $1+\epsilon$, but only generically. Indeed, given $n\ge 3$ let X be the hyperplane $x_0+\dots+x_{n-1}=0$ in $\mathbb{P}^{n-1}$, and let D be the restriction to X of the sum of the coordinate hyperplanes. Then X is smooth and D has normal crossings. If $x_0,\dots,x_{n-1}$ are nonzero integers satisfying $x_0+\dots+x_{n-1}=0$ and $(x_0,\dots,x_{n-1})=1$, then $P:=[x_0:\dots:x_{n-1}]\in X$, $h(P)=\log\max\{|x_0|,\dots,|x_{n-1}|\}$, and $N^{(1)}(D,P)=\log\prod_{p|x_0\cdots x_{n-1}}p$. Then Conjecture 2.3 would imply that

$$\max\{|x_0|,\dots,|x_{n-1}|\}\le C\prod_{p|x_0\cdots x_{n-1}}p^{1+\epsilon}$$

for all $x_0,\dots,x_{n-1}$ as above *outside a proper Zariski-closed subset.*

This subset is, in fact, essential: consider the following example. It is well known that there exist infinitely many triples $(a,b,c)\in\mathbb{Z}^3$ with $a+b+c=0$, $(a,b,c)=1$, and $\max\{|a|,|b|,|c|\}\ge\prod_{p|abc}p$. For such triples, we have

$$a^2+2ab+b^2-c^2=0\ .$$

Letting $a'=a^2$, $b'=2ab$, $c'=b^2$, and $d'=-c^2$, we have

$$\begin{aligned} a'+b'+c'+d'&=0\ ,\\ (a',b',c',d')&=1\ ,\\ h([a':b':c':d'])&=2h([a:b:c])+O(1)\ , \text{ and}\\ \prod_{p|a'b'c'd'}p&=\prod_{p|abc}p\ . \end{aligned}$$

Thus there are infinitely many points with

$$\max\{|a'|,|b'|,|c'|,|d'|\}\gg\prod_{p|a'b'c'd'}p^2\ .$$

This does not contradict Conjecture 2.3, however, because of the exceptional subset Z. Instead, it shows that working with Z is the hardest part in determining what the conjectural exponent should be.

Even with an exceptional subset, and requiring the x_i to be pairwise relatively prime, though, it is fairly easy to see that an exponent better than 1 is not possible:

Proposition 2.6. *Let $n \ge 3$ be an integer and let $\epsilon > 0$ be arbitrary. Then there exists a Zariski-dense set of points on the hyperplane $x_0 + \dots + x_{n-1}$ in $\mathbb{P}^{n-1}$ having pairwise relatively prime homogeneous coordinates $[x_0 : \dots : x_{n-1}]$ such that*

$$\prod_{p \mid x_0 \cdots x_{n-1}} p \le \epsilon \max\{|x_0|, \dots, |x_{n-1}|\} .$$

Proof. For $n = 3$ this is already well known.

Assume for now that $n = 4$. Since $\log_9 25$ is irrational, its positive integer multiples are dense in $\mathbb{R}/2\mathbb{Z}$; hence there exist positive integers e_1 and e_2 with e_1 odd, such that $0 < 9^{e_1} - 25^{e_2} < \epsilon \cdot 9^{e_1}$. Let $x_1 = -9^{e_1}$, $x_2 = 25^{e_2}$, $x_3 = 1$, and choose x_0 so that the sum vanishes. This gives a tuple (x_0, x_1, x_2, x_3) whose elements are pairwise relatively prime, whose elements add up to zero, and which satisfies

$$\prod_{p \mid x_0 x_1 x_2 x_3} p \le 15|x_0| \le 15\epsilon \max\{|x_0|, |x_1|, |x_2|, |x_3|\} .$$

The set of all such points is Zariski-dense in the hyperplane $x_0 + x_1 + x_2 + x_3 = 0$: otherwise, some irreducible curve would contain infinitely many of them, and hence some irreducible polynomial $f(X, Y)$ would satisfy $f(9^{e_1}, 25^{e_2}) = 0$ for infinitely many pairs (e_1, e_2). Applying the unit equation for $\mathbb{Z}[1/15]$ to the terms of such equations gives finitely many linear relations in those terms, leading to a contradiction.

Finally, assume that $n \ge 5$. Let $p_1, \dots, p_{n-1}$ be distinct primes greater than n, let $r_1, \dots, r_{n-1}$ be positive integers such that $p_i^{r_i} \equiv 1 \pmod{p_j}$ for all $i \ne j$, and let $q_i = p_i^{r_i}$ for all i. As before, there exist infinitely many pairs (e_1, e_2) of positive integers such that

$$0 < q_1^{e_1} - q_2^{e_2} < \epsilon q_1^{e_1} .$$

For such pairs, let $e_3, \dots, e_{n-1}$ be nonnegative integers such that

$$q_3^{e_3} + \dots + q_{n-1}^{e_{n-1}} < q_1^{e_1} - q_2^{e_2} ,$$

let x_0 be the difference, let $x_1 = -q_1^{e_1}$, and let $x_i = q_i^{e_i}$ for $i = 2, \dots, n-1$. Then one will have $x_0 + \dots + x_{n-1} = 0$, the x_i will be pairwise relatively prime, and the inequality

$$\prod_{p \mid x_0 \cdots x_{n-1}} p \le p_1 \cdots p_{n-1} |x_0| \le p_1 \cdots p_{n-1} \epsilon \max\{|x_0|, \dots, |x_{n-1}|\}$$

will hold. Moreover, these points will lie outside any given proper Zariski-closed subset: if e_1 and e_2 are large, then $q_1^{e_1} - q_2^{e_2}$ will be large (*e.g.*, by Baker's theorem), and then there will be enough choices for each of the remaining e_i to avoid the subset. □

Finally, we mention a conjecture of A. Buium [**B**]:

Conjecture 2.7. *Let A be an abelian variety over k, let D be an ample effective divisor on A, and let $\mathscr{A}$ be an ample line sheaf on A. Then $N^{(1)}(D,P) \gg h_D(P)+O(1)$ for all $P \in A(k) \setminus D$.*

This, too, follows from Conjecture 2.3. Indeed, let X be a closed subvariety of A not contained in the support of D. There exists a proper birational morphism $\pi\colon X' \to X$ such that X' is nonsingular and $D' := (\pi^* D)_{\mathrm{red}}$ is a normal crossings divisor on X'. Rational points $P \in X(k)$ lying outside a proper Zariski-closed subset Z may be lifted to $P' \in X'(k)$; for these points Conjecture 2.3 gives

$$\begin{aligned} N^{(1)}(D,P) &= N^{(1)}(D',P') + O(1) \\ &\ge h_{\mathscr{K}'(D')}(P') - \epsilon\, h_{\mathscr{A}'}(P') - O(1) \end{aligned}$$

(after enlarging Z). Here $\mathscr{K}'$ is the canonical line sheaf on X' and $\mathscr{A}'$ is a big line sheaf on X'. But X has Kodaira dimension ≥ 0 and D is ample, so $\mathscr{K}'(D')$ is big; hence (after enlarging Z again) we have

$$h_{\mathscr{K}'(D')}(P') - \epsilon\, h_{\mathscr{A}'}(P') \gg h_D(P) + O(1)$$

for suitable $\epsilon > 0$. Starting with $X = A$ and repeating with smaller and smaller X coming from the exceptional Zariski-closed subset, we then conclude by noetherian induction that Conjecture 2.7 follows from Conjecture 2.3.

§3. The harder implication

The goal of this section is to prove:

Theorem 3.1. *Conjecture 2.1 implies Conjecture 2.3.*

The general strategy of the proof is to start with the data X, D, etc. for which one wants to derive (2.3.1), determine a large integer e depending on these data, and construct a cover X' of X, ramified to order $\ge e$ everywhere above D and unramified elsewhere. Then a point $P \in X(\bar{k})$ of bounded degree will lift to a point $P' \in X'(\bar{k})$ of bounded (but larger) degree, and the ramification of $k(P')$ over $k(P)$ will occur only at places contributing to $N(D,P)$. But the contribution to $d(P') - d(P)$ at a place w of $k(P')$ is limited, so in fact $d(P') - d(P)$ is more closely related to $N^{(1)}(D,P)$. One can then apply Conjecture 2.1 on X' to deduce Conjecture 2.3 for points P on X outside a proper Zariski-closed subset.

This general method of proof has been used previously by the author ([**V**], pp. 71–72), and by Darmon and Granville [**D-G**].

We start with a lemma describing how $K_X + D$ changes when pulled back via a generically finite morphism.

Lemma 3.2. *Let $\pi\colon X' \to X$ be a generically finite morphism of smooth varieties, and let D and D' be normal crossings divisors on X and X', respectively, such that*

$\operatorname{Supp} D' = (\pi^* D)_{\text{red}}$. Let $\mathscr{K}$ and $\mathscr{K}'$ be the canonical line sheaves on X and X', respectively. Then

$$\mathscr{K}'(D') \ge \pi^*(\mathscr{K}(D))$$

relative to the cone of line sheaves with $h^0 > 0$. Moreover, $\mathscr{K}'(D') \otimes \pi^*(\mathscr{K}(D))^\vee$ has a global section vanishing only on the support of D' or where π ramifies.

Proof. We have a natural map $\pi^*\Omega^1_X[\log D] \to \Omega^1_{X'}[\log D']$ which is an isomorphism at generic points of X'; hence taking the maximal exterior power gives an injection of sheaves $\pi^*(\mathscr{K}(D)) \to \mathscr{K}'(D')$. See also ([**I**], §11.4a). □

Consequently, one may use Chow's lemma and resolution of singularities to find a smooth projective variety X' and a proper birational morphism $\pi\colon X' \to X$ such that $D' := (\pi^* D)_{\text{red}}$ has normal crossings; the lemma then shows that

$$h_{\mathscr{K}'(D')} \ge h_{\mathscr{K}(D)} \circ \pi + O(1)$$

outside of a proper Zariski-closed subset. Since $\operatorname{Supp} D' = \pi^{-1}(\operatorname{Supp} D)$, we have $N^{(1)}(D', P) = N^{(1)}(D, \pi(P)) + O(1)$ for $P \in X'(\bar{k})$; hence Conjecture 2.3 for X' and D' implies the same conjecture for X and D. Thus we may assume that X is projective.

Next, we note that if D and D_1 are effective divisors such that $D + D_1$ has normal crossings, then Conjecture 2.3 for $D + D_1$ implies that the conjecture holds also for D. Indeed,

$$\begin{aligned} N^{(1)}(D + D_1, P) - N^{(1)}(D, P) &\le N^{(1)}(D_1, P) \\ &\le N(D_1, P) \\ &\le h_{\mathscr{O}(D_1)}(P) + O(1)\ , \end{aligned}$$

so (2.3.1) with D replaced by $D + D_1$ implies the original (2.3.1).

Thus, we may assume that D is very ample; in addition, we will enlarge D at one point in the proof.

Lemma 3.3. *Let $e \in \mathbb{Z}_{>0}$ and let D_1 be an effective divisor such that $D_1 \sim D$ and $D + D_1$ has normal crossings. Then there is a smooth variety X_1 and a proper generically finite morphism $\pi_1\colon X_1 \to X$ such that the support of the ramification divisor of π_1 is equal to $\pi_1^{-1}(\operatorname{Supp}(D + D_1))$, all components of $\pi^*(D + D_1)$ have multiplicity $\ge e$ unless they lie over $D \cap D_1$, $(\pi^*(D + D_1))_{\text{red}}$ has normal crossings, and $K(X_1) = K(X)(\sqrt[e]{f})$ for some $f \in K(X)^*$.*

Proof. Pick $f \in K(X)^*$ such that $(f) = D - D_1$. Let $\pi_0\colon X_0 \to X$ be the normalization of X in the field $K(X)(\sqrt[e]{f})$, let $\pi\colon X_1 \to X_0$ be a desingularization of X_0 such that $\left(\pi^*(\pi_0^*(D + D_1))\right)_{\text{red}}$ has normal crossings, and let $\pi_1 = \pi_0 \circ \pi$. We may assume that π_1 is étale outside of $\pi_0^{-1}(\operatorname{Supp} D)$. If $P \notin D \cap D_1$, then there is an open neighborhood U of P such that $D\big|_U = (f)$ and $D_1\big|_U = 0$, or $D\big|_U = 0$ and

$D_1\big|_U = (1/f)$. Then $\pi_0^*(D+D_1)$ over U is e times the (principal) divisor $(\sqrt[e]{f})$ or $(1/\sqrt[e]{f})$. Therefore all components of $\pi_1^*(D+D_1)$ meeting $\pi_1^{-1}(U)$ have multiplicity divisible by e . □

Lemma 3.4. *There exists a normal crossings divisor D^* on X such that*

(i). *D^*-D is effective;*
(ii). *for all $e\in\mathbb{Z}_{>0}$ there exists a smooth variety X' and a proper generically finite morphism $\pi\colon X'\to X$ such that $(\pi^*D)_{\mathrm{red}}$ has normal crossings and all components of π^*D have multiplicity $\ge e$ (or zero);*
(iii). *$\pi\colon X'\to X$ is unramified outside of $\pi^{-1}(\operatorname{Supp} D)$; and*
(iv). *$K(X')=K(X)(\sqrt[e]{f_1},\dots,\sqrt[e]{f_n})$ for some $f_1,\dots,f_n\in K(X)^*$.*

Proof. Let $n=\dim X$, and let $D_1,\dots,D_n$ be effective divisors on X such that $D_i\sim D$ for all i , $D^*:=D+D_1+\dots+D_n$ has normal crossings, and $D\cap D_1\cap\dots\cap D_n=\emptyset$. (Such divisors exist by Bertini's theorem.)

Now pick $e\in\mathbb{Z}_{>0}$. For $i=1,\dots,n$ let $\pi_i\colon X_i\to X$ be as in Lemma 3.3, applied to the divisors $D_i\sim D$. Let X' be a desingularization of the normalization of X in the compositum $K(X_1)\cdots K(X_n)$ such that, if $\pi\colon X'\to X$ is the obvious morphism, then $(\pi^*D^*)_{\mathrm{red}}$ has normal crossings and such that X' dominates $X_1,\dots,X_n$. We may also assume that π is étale outside of $\pi^{-1}(\operatorname{Supp} D)$. Pick a component E of π^*D^* . There exists some i such that $\pi(E)\subseteq D\cup D_i$ but $\pi(E)\nsubseteq D\cap D_i$; then the image of E in X_i is contained in a component of $\pi^{-1}(D+D_1)$ of multiplicity $\ge e$. □

After replacing D with D^* , we have the following situation: D has normal crossings, X' is smooth, $D':=(\pi^*D)_{\mathrm{red}}$ has normal crossings, all components of π^*D have multiplicity $\ge e$, and $\pi\colon X'\to X$ is unramified outside of D' .

The methods of ([**K**], Thm. 17) and ([**A-K**], Lemma 5.8) allow one to construct such a π that is finite, but this is not necessary for the present argument.

The key step in the proof of Theorem 3.1 is the following lemma, which gives a bound on $d_k(P')-d_k(P)$.

Lemma 3.5. *Pick models $\mathscr{X}$ and $\mathscr{X}'$ for X and X' , respectively, for which the divisors D and D' extend to effective divisors on $\mathscr{X}$ and $\mathscr{X}'$, respectively, and π extends to a morphism $\pi\colon\mathscr{X}'\to\mathscr{X}$. Suppose S contains all places lying over e , all places of bad reduction for $\mathscr{X}$ and $\mathscr{X}'$, all places where D and D' have vertical components, and all places where π is ramified outside the support of D' . Let $r\in\mathbb{Z}_{>0}$. For points $P\in X(\bar k)\setminus\operatorname{Supp} D$ of degree $\le r$ over k and points $P'\in\pi^{-1}(P)$, we have*

$$d_k(P')-d_k(P)\le N^{(1)}(D,P)-N(D',P')+\frac{1}{e}h_D(P)+O(1)\ , \tag{3.5.1}$$

where the constant in $O(1)$ depends only on X , D , e , X' , π , r , and S .

Proof. By (1.1) we have

$$d_k(P')-d_k(P)=d_{k,S}(P')-d_{k,S}(P)+O(1)\ . \tag{3.5.2}$$

Let $E = k(P)$ and $E' = k(P')$. For places $w' \in M_{E'}$ lying over places $w \in M_E$, let $e_{w'/w}$ denote the index of ramification of w' over w. By (1.2), we then have

$$d_{k,S}(P') - d_{k,S}(P) = \sum_{w' \in M_{E'} \setminus S} \frac{e_{w'/w} - 1}{[E' : k]} \mu(K(w'))\ .$$

If the closure in $\mathscr{X}'$ of P' does not meet D' at w', then E' is unramified over E at w'; if it does meet, then $e_{w'/w} \le e$ because E' is generated over E by e^{th} roots of elements of E (by condition (iii) in Lemma 3.4). Thus

$$d_{k,S}(P') - d_{k,S}(P) \le (e-1)N^{(1)}(D', P')\ .$$

Combining this with (3.5.2) then gives

$$d_k(P') - d_k(P) \le (e-1)N^{(1)}(D', P') + O(1)\ . \tag{3.5.3}$$

Now consider the right-hand side of (3.5.1). Since all components of $\pi^* D$ have multiplicity $\ge e$,

$$N^{(1)}(D, P) - N^{(1)}(D', P') \ge (e-1)N^{(1)}(D', P')\ .$$

Also,

$$\begin{aligned} N(D', P') - N^{(1)}(D', P') &\le N(D', P') \\ &\le \frac{1}{e} N(D, P) \\ &\le \frac{1}{e} h_D(P) + O(1)\ . \end{aligned}$$

Combining these two inequalities then gives

$$N^{(1)}(D, P) - N(D', P') + \frac{1}{e}\, h_D(P) \ge (e-1)N^{(1)}(D', P') + O(1)\ .$$

Comparing this with (3.5.3) then gives (3.5.1). □

To prove the theorem, it remains only to assemble the ingredients.

Proof of Theorem 3.1. By Kodaira's lemma we may assume (after adjusting ϵ) that $\mathscr{A}$ is ample. Then

$$h_D \le c\, h_{\mathscr{A}} + O(1) \tag{3.6}$$

for some constant c depending only on X, D, and A. Pick $e \ge c/\epsilon$, and let X' be a generically finite cover of X as in Lemma 3.4. Let D and D' be as discussed following that lemma, and enlarge S so that the conditions of Lemma 3.5 hold. This will ultimately give an inequality (2.3.1) relative to this larger set S, which trivially implies the same inequality for the original S. Points $P\in X(\bar k)\setminus\operatorname{Supp}D$ of bounded degree lift to points $P'\in X'(\bar k)$, also of bounded degree. We now show that Conjecture 2.1 applied to X' and D' implies Conjecture 2.3 for D and X. By the former conjecture (using (2.2)), we have

$$N(D',P')+d_k(P)\ge h_{\mathscr{K}'(D')}(P')-\epsilon' h_{\mathscr{A}'}(P')-O(1)\;, \tag{3.7}$$

provided $P'\notin Z'$, where Z' is a proper Zariski-closed subset (depending also on ϵ' and $\mathscr{A}'$); here $\mathscr{A}'$ is a big line sheaf on X'. We want to show that (3.7) implies Conjecture 2.3; *i.e.*,

$$N^{(1)}(D,P)+d_k(P)\ge h_{\mathscr{K}(D)}(P)-\epsilon\, h_{\mathscr{A}}(P)-O(1)\;. \tag{3.8}$$

By Lemma 3.2, $\mathscr{K}'(D')\otimes\pi^*(\mathscr{K}(D))^\vee$ has a global section vanishing nowhere on $X'\setminus\operatorname{Supp}D'$; hence, by functoriality of heights and positivity of heights relative to effective divisors,

$$h_{\mathscr{K}'(D')}(P')\ge h_{\mathscr{K}(D)}(P)-O(1) \tag{3.9}$$

for all $P\in X'(\bar k)\setminus D'$.

By Lemma 3.5,

$$N(D',P')+d_k(P')\le N^{(1)}(D,P)+d_k(P)+\frac1e\,h_D(P)+O(1)\;. \tag{3.10}$$

Let $\mathscr{A}'=\pi^*\mathscr{A}$; this is big on X'; also choose $\epsilon'>0$ such that $\epsilon'<\epsilon-c/e$. By (3.6) we then have

$$\frac1e\,h_D(P)+\epsilon' h_{\mathscr{A}'}(P')\le\epsilon\, h_{\mathscr{A}}(P)+O(1)\;. \tag{3.11}$$

Thus, (3.9)–(3.11), combined with (3.7), imply that (3.8) holds if P lies outside the proper Zariski-closed subset $Z:=\pi(Z')$. □

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