Differential algebraic geometry and abc

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Lecture 1. Motivation: abc on affine varieties.

We start by explaining a generalization of **abc** that makes sense for any affine variety over a global field. The usual **abc** is simply the case of $\mathbf{P}^1\backslash\{0,1,\infty\}$. We then state our main result that says that this generalized **abc** holds, under certain trace conditions, for affine open sets of abelian varieties over function fields.

Let K be a field equipped with a family of absolute values $|\ |_v : K \to [0,\infty)$, $v \in M$, all of which, except finitely many, are non-archimedian. Set $v(x) = -log|x|_v$ for $x \in K^{\times}$. Assume (m_v) is a collection of positive integers such that the "product formula"

$$\sum_v m_v v(x) = 0, \quad x \in K^{\times}$$

holds. Set

$$\gamma_v := \inf\{v(K^{\times}) \cap (0,\infty)\}$$

Also, for any $\eta = (\eta_1,...,\eta_N) \in K^N$ and $v \in M$ set

$$v(\eta) = \min_j v(\eta_j)$$

Define the (affine, logarithmic) height

$$height_{\mathbf{A}^N} : \mathbf{A}^N(K) = K^N \to [0,\infty)$$

by the formula

$$height_{\mathbf{A}^N}(\eta) = -\sum_{v(\eta)\leq 0} m_v v(\eta) = \sum_v m_v \max_j\ \log^+ |\eta_j|_v$$

where $\log^+ x := \max\{\log x, 0\}$, $x \in [0,\infty)$. Note that

$$height_{\mathbf{A}^N}(\eta) = height_{\mathbf{P}^N}(1 : \eta_1 : ... : \eta_N)$$

where

$$height_{\mathbf{P}^N}(x_0 : ... : x_N) = \sum_v m_v \max_j \log |x_j|_v$$

is the usual height in projective space. On the other hand define the (logarithmic) conductor

$$cond_{\mathbf{A}^N} : \mathbf{A}^N(K) = K^N \to [0, \infty)$$

by the formula

$$cond_{\mathbf{A}^N}(\eta) = \sum_{v(\eta) \leq 0} m_v \gamma_v$$

Clearly, by the very definition of γ_v we have

$$cond_{\mathbf{A}^N}(\eta) \leq height_{\mathbf{A}^N}(\eta), \quad \eta \in K^N$$

We need the following piece of notation. Let $f, g : S \to [0, \infty)$ be two real functions on a set S; we write

$$f \leq g + O(1)$$

if there exists a positive real constant C such that

$$f(P) \leq g(P) + C, \quad P \in S$$

and we write $f = g + O(1)$ if $f \leq g + O(1)$ and $g \leq f + O(1)$. We write

$$f << g + O(1)$$

if there exist two real positive constants C_1, C_2 such that

$$f(P) \leq C_1 g(P) + C_2, \quad P \in S$$

and we write $f \equiv g + O(1)$ if $f << g + O(1)$ and $g << f + O(1)$. Coming back to *height* and *cond* one easily checks that if $P : \mathbf{A}^N(K) \to \mathbf{A}^n(K)$ a map given by an $n-$tuple of polynomials in N variables with $K-$coefficients then we have

$$height_{\mathbf{A}^n} \circ P << height_{\mathbf{A}^N} + O(1)$$

$$cond_{\mathbf{A}^n} \circ P \leq cond_{\mathbf{A}^N} + O(1)$$

This permits to define the **height** and the **conductor** for any affine variety as follows. Let U be an affine variety over K. Let $i : U \to \mathbf{A}^N$ be a closed immersion and define

$$height_U : U(K) \to [0, \infty)$$

$$cond_U : U(K) \to [0, \infty)$$

by the formulae

$$height_U(P) := height_{\mathbf{A}^N}(i(P)), \quad P \in U(K)$$

$$cond_U(P) := cond_{\mathbf{A}^N}(i(P)), \quad P \in U(K)$$

By the above discussion, if $height_U$ and $cond_U$ are defined by a closed immersion i and $height'_U$ and $cond'_U$ are defined by a closed immersion i' then

$$height_U \equiv height'_U + O(1)$$

$$cond_U = cond'_U + O(1)$$

In particular we have

$$cond_U << height_U + O(1)$$

Here is our basic definition.

Definition. *We say that the* **abc** *estimate holds on U if*

$$height_U << cond_U + O(1)$$

In this lecture we would like to understand what are, conjecturally, the affine varieties U on which the **abc** estimate holds. To tackle this question, and establish the link with the "usual" **abc** we need to be more specific about our field K. Assume, in what follows, that we are in one of the following situations:

1) Number field case. K is a number field equipped with its standard family of absolute values $(| \ |_v)$ (normalized in such a way that they extend the standard absolute values of $\mathbf{Q}$; in particular, if v divides a rational prime

p, then $|p|_v = p^{-1}$.) We take $m_v = [K_v : \mathbf{Q}_v]$. So, for any non-archimedian v, dividing an unramified rational prime p, we have $\gamma_v = v(p) = \log p$ so $m_v\gamma_v = \log \mathbf{N}v$ where $\mathbf{N}v$ is the norm of v (cardinality of the residue field of v). Since there are only finitely many ramified primes, we have

$$cond_{\mathbf{A}^N}(\eta) = \sum_{v(\eta)<0} \log \mathbf{N}v, \quad \eta \in K^N$$

2) Function field case. K is a function field of one variable over an algebraically closed field k of characteristic zero. We equip K with the absolute values $|\ |_v$ arrising from the $k-$rational points v of the smooth projective model V of K/k; we normalise them via the condition $\gamma_v = 1$ for all v, and we take $m_v = 1$ for all v. So in this case $cond_{\mathbf{A}^N}(\eta)$, for $\eta \in K^N$, is simply the number of points $v \in V$ which are poles for at least one of the rational functions η_j.

It is a trivial exercise to show that if, say, $K = \mathbf{Q}$ or $K = k(t)$, and if there exists a non-constant morphism of $K-$varieties

$$\mathbf{P}^1 \backslash \{0, \infty\} \to U$$

into an affine $K-$variety U then the **abc** estimate fails on U. On the other hand the optimist would be tempted to believe that the presence of such morphisms is the only obstruction to the **abc** estimate; we make, optimistically, the following:

Conjecture. *Assume U is a smooth affine variety over K and assume that any morphisms of $K-$varieties $\mathbf{P}^1\backslash\{0,\infty\} \to U$ is constant. Then the* **abc** *estimate holds on U.*

Our Conjecture should be viewed as an afine analogue of "Lang's conjecture" saying that if X is a smooth **projective** variety over K and if any morphism from an algebraic group G to X is constant then the points of $X(K)$ have bounded height; hence, in the number field case $X(K)$ is finite. (In Lang's conjecture one allows morphisms $G \to X$ defined over the algebraic closure $\bar{K}$ of K; it might be reasonable to allow this in our Conjecture as well.)

In case $K = \mathbf{Q}$ and $U = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, the Conjecture above simply says that the **abc** estimate holds on this particular U; note that this is equivalent to the following:

Variant of abc conjecture. *For any relatively prime integers a, b, c with $a + b = c$ there exist real numbers $C > 1$ and $\mu > 1$ such that*

$$\max\{|a|, |b|, |c|\} \leq C \cdot rad(abc)^{\mu}$$

(As usual $rad(n)$, where n is an integer, is defined as the product of all primes dividing n; this Variant is weaker than the **abc** of Masser and Oesterlé which predicts that μ can be made as close to 1 as we want. However this Variant still implies, say, the assymptotic Fermat.)

To see the equivalence between the Conjecture and the Variant of **abc** for $K = \mathbf{Q}$ and $U = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, embed U into $\mathbf{A}^2$ via the map

$$(1 : x) \mapsto \left(x, \frac{1}{x(x-1)} \right)$$

Then, for coprime integers a, b, c with $a + b = c$, we have that $height_U(1 : \frac{c}{a})$ equals

$$height_{\mathbf{P}^2}(1 : \frac{c}{a} : \frac{a^2}{cb}) = \log \max(|abc|, |b^2 c|, |a^3|) \equiv \log \max(|a|, |b|, |c|) + O(1)$$

On the other hand

$$cond_U(1 : \frac{c}{a}) = cond_{\mathbf{A}^2}(\frac{c}{a}, \frac{a^2}{cb}) = \log \ rad(abc)$$

Note that, by a result due (independently) to Mason and Silverman, the **abc** estimate holds for the projective line minus 3 points. On the other hand results of Voloch [V], Brownawell-Masser [BM], and Wang [W] can suitably be interpreted as **abc** estimates "outside some exceptional loci" for projective spaces minus unions of hyperplanes.

Theorem. *(**abc** for abelian varieties with trace zero [B94']) In the function field case the **abc** estimate holds for any affine open set of an abelian variety with trace zero.*

Theorem. *(**abc** for isotrivial abelian varieties [B98]) Let K be a function field over k, let A_k be an abelian variety over k and D_k a divisor in A which does not contain any translate of a non-zero abelian subvariety (in particular D_k is ample so $U_k := A_k \backslash D_k$ is affine). Let $U := U_k \otimes_k K$. Then the **abc** estimate holds on U.*

The above Theorems are immediate consequences of the following stronger results:

Theorem. *(Bounded Multiplicity Theorem, trace zero case [B94']) Assume A/K is an abelian variety with trace zero and let $f \in K(A)$ be a rational function. Then there exists a constant C depending only on K, A, f with the following property. For any point $P \in A(K)$ where f is defined and does not vanish, all zeroes and poles of $f(P) \in K^\times$ have multiplicity at most C.*

Theorem. *(Bounded Multiplicity Theorem, isotrivial case [B98]) Let X be a smooth projective curve over k, A an abelian variety over k, and D an effective divisor on A. Assume that D contains no translate of a non zero abelian subvariety. Then there exists a real constant $C > 0$, depending only on X, A, and D with the property that for any morphism $f : X \to A$, with $f(X) \not\subset D$, all points of the divisor f^*D have multiplicity at most C.*

The proofs of the results above are based on "differential algebraic geometry". A characteristic p version of these results was proved by Scanlon [Sc].

Lecture 2. Differential algebraic geometry.

Let $\mathcal{F}$ be a field of characteristic zero equipped with a derivation δ. We let $\mathcal{C}$ be the constant field. (If K is a function field we will always assume we have fixed a non-zero $k-$derivation on it and (K, d) is embedded into $(\mathcal{F}, \delta)$.)

Following the classical work of Ritt and Kolchin one defines the $\delta-$polynomial ring $\mathcal{F}\{T\}$, where T is an $n-$tuple of variables, as the (usual) polynomial ring over $\mathcal{F}$ in the variables $T^{(i)}$, $i \geq 0$. The *order* of $A \in \mathcal{F}\{T\}$ is the highest i such that a variable $T_j^{(i)}$ is present in A. $\mathcal{F}$ is called $\delta-$closed if for any $A, B \in \mathcal{F}\{T\}$, T a variable, such that $ord\ B < ord\ A$, there exists $a \in \mathcal{F}$ such that $A(a) = 0$ and $B(a) \neq 0$.

We need more definitions. A $D-$scheme is simply an $\mathcal{F}-$scheme V with a given derivation on $\mathcal{O}_V$ that lifts δ. $D-$schemes form a category (morphisms are required to commute with the derivations). A $D-$group scheme is a $D-$scheme which is also an $\mathcal{F}-$group scheme such that the multiplication, inverse, and unit are morphisms of $D-$schemes. A $D-$variety is a $D-$scheme which is also a variety over $\mathcal{F}$. An algebraic $D-$group is a $D-$group scheme which is also an algebraic group over $\mathcal{F}$.

One has a forgetful functor

$$\{D - schemes\} \rightarrow \{\mathcal{F} - schemes\}, V \mapsto V^!$$

It has a right adjoint

$$\{\mathcal{F} - schemes\} \rightarrow \{D - schemes\}, X \mapsto X^\infty$$

$$Hom_{\mathcal{F}}(V^!, X) = Hom_D(V, X^\infty)$$

defined as follows. If $X = Spec\ \mathcal{F}[T]/I$ set $X^\infty := Spec\ \mathcal{F}\{T\}/[I]$, $[I] := (I, \delta I, \delta^2 I, ...)$. In the non-affine case one glues the affine pieces. (Note that X^∞ is the inverse limit of a system X^n of varieties obtained by truncating everything to order n. One has a natural map $\nabla : X(\mathcal{F}) \rightarrow X^\infty(\mathcal{F})$ which in coordinates sends $a \in \mathcal{F}$ into $(a, \delta a, \delta^2 a, ...)$. Pull backs via this map of Zariski closed sets are called $\delta-$closed sets. (Kolchin Topology). If $\mathcal{F}$ is $\delta-$closed there is a bijection between delta closed sets and reduced closed $D-$subschemes of X^∞. If $\Sigma \subset X(\mathcal{F})$ is $\delta-$closed, corresponding to a $D-$subscheme $H \subset X^\infty$ then one defines the absolute dimension $a(\Sigma)$ as the maximum of the transcendence degrees over $\mathcal{F}$ of the function fields of the irreducible components of H. Example: $a(X(\mathcal{F})) = \infty$ if $dim\ X > 0$. If X descends to $\mathcal{C}$ (i.e. comes from a variety $X_{\mathcal{C}}$ over $\mathcal{C}$) then $a(X_{\mathcal{C}}(\mathcal{C})) = dim\ X$. For G an algebraic group over $\mathcal{F}$, the above bijection induces a bijection between the $\delta-$closed subgroups of finite absolute dimension of $G(\mathcal{F})$ and algebraic D-subgroups of G^∞.

By a $\delta-$function on $X(\mathcal{F})$, where X is a variety, we mean a map $X(\mathcal{F}) \to \mathcal{F}$ obtained by composing ∇ with a regular map on X^∞. A $\delta-$character on an algebraic group will mean a $\delta-$function which is also an additive homomorphism.

Theorem. *($\delta-$density Theorem [B93]) If X is a smooth projective unirational $\mathcal{F}-$variety then any $\delta-$function on $X(\mathcal{F})$ is constant. Moreover, if X defined over an intermediate field L between $\mathcal{C}$ and $\mathcal{F}$, $L \neq \mathcal{C}$, then $X(L)$ is $\delta-$dense in $X(\mathcal{F})$.*

Theorem. *(Finiteness of absolute dimension [B92]) Let A be an abelian $\mathcal{F}-$variety of dimension g. Then the intersection $A^\sharp$ of the kernels of all $\delta-$characters of A has finite absolute dimension $g \leq a(A^\sharp) \leq 2g$. Consequently, the $\delta-$closure of any finite rank subgroup of $A(\mathcal{F})$ has finite absolute dimension $\leq 2g + r$, where r is the rank. In particular, in the trace zero case, the $\delta-$closure of the group of division points of $A(K)$ has finite absolute dimension.*

The $\delta-$characters are the incarnation, in differential algebraic geometry, of the "Manin homomorphisms"; but our finiteness result ($a(A^\sharp) < \infty$) is quite different in nature from the Manin-Chai "Theorem of the kernel". What the latter says, in the trace zero case, in our terminology is the following:

Theorem of the Kernel. *[Manin-Chai] Let A be an abelian $\mathcal{F}-$variety with $\mathcal{F}/\mathcal{C}-$trace zero. Then for any intermediate $\delta-$field $\mathcal{C} \subset L \subset \mathcal{F}$ of definition for A which is finitely generated over $\mathcal{C}$ we have $A^\sharp \cap A(L) = A(L)_{tors}$.*

Theorem. *($\delta-$maps on curves [B94]) Let X be a smooth projective curve over $\mathcal{F}$ of genus $g \geq 2$, that does not descend to $\mathcal{C}$. Then there exists an injective $\delta-$map $\phi : X \to \mathbf{A}^n$.*

In a certain precise sense one can actually prove more namely that projective curves of genus at least 2 are affine in this geometry.

Half way towards our **abc** for abelian varieties we have the following differential algebraic generalisation of the geometric Lang conjecture on subvarieties of abelian varieties:

Theorem. *(Differential Algebraic Lang [B92]) Let A be an abelian variety over $\mathcal{F}$ with trace zero over $\mathcal{C}$. Let $\Sigma \subset A(\mathcal{F})$ be a $\delta-$closed subgroup of finite absolute dimension and $X \subset A$ a closed subvariety. Then there exist in X finitely many translates of abelian subvarieties whose union contains $X(\mathcal{F}) \cap \Sigma$.*

The above Theorem is indeed a generalisation of the geometric Lang conjecture because, together with the Theorem on finite absolute dimension, it formally implies the

Theorem. *(Geometric Lang Conjecture [B92].) Let A be an abelian variety over K with trace zero over k. Let $\Gamma \subset A(\bar{K})$ be a subgroup of finite rank and $X \subset A$ a closed subvariety. Then there exist in $X(\bar{K})$ finitely many translates of abelian subvarieties whose union contains $X(\bar{K}) \cap \Gamma$.*

Lecture 3. Description of proofs.

a. Descent results.

An $\mathcal{F}-$variety (resp. an algebraic group over $\mathcal{F}$) is said to descend to constants if it comes, via base change, from a variety (resp. an algebraic group) over $\mathcal{C}$.

The next four theorems are proved via complex analytic arguments; for the first theorem Gillet showed me an argument based on formal schemes which should be also considered in some sense analytic; this kind of argument, in formal geometry, does not seem to apply to the other results.

Theorem. *(Descent of projective varieties [B87]) Any projective $D-$variety descends to constants.*

Theorem. *(Descent of linear algebraic groups [B92]) Any linear algebraic D-group descends to constants.*

Remark: the above fails for non-linear groups. A VERY interesting question: Let G be an algebraic D-group and H be its maximum connected linear algebraic subgroup; does H descend to constants ?

We won't need the previous two Theorems for our diophantine purposes. For the **abc** Theorem for abelian varieties with trace zero we will need:

Theorem. *(Descent via D-groups [B92]) Let G be an algebraic D-group, $V \subset G$ a closed $D-$subvariety, and $V \to W$ a dominant morphism to a projective variety W of general type. Then the Albanese variety $Alb(W)$ descends to constants.*

For the **abc** Theorem for isotrivial abelian varieties we need:

Theorem. *(Descent via D-groups, split case [B98]) Let W be a projective variety of general type over K. Assume W is a closed subvariety of A_K, where A is an abelian $k-$variety. Let G be any algebraic $D-$group, $V \subset G$ a $D-$subvariety and $u : V \to W$ be a dominant morphism. Then, after replacing K by a finite extension of it, one may find a closed $k-$subvariety $Z \subset A$ and a point $Q \in A(K)$ such that $W = Z_K + Q$ in A_K. Moreover, if we view W as a $D-$scheme by trivially lifting δ from K to $W \simeq Z_K = Z \otimes_k K$, then $u : V \to W$ is necessarily a morphism of $D-$schemes.*

Let us sketch the proof of the Theorem on "descent via D-groups". We may assume $\mathcal{C}$ is the field of complex numbers. We may further reduce ourselves to the case when $(V, W, G, Spec\ \mathcal{F}, \delta)$ comes, via base change, from a "situation in complex algebraic geometry" $(\mathbf{V}, \mathbf{W}, \mathbf{G}, \mathbf{S}, \delta_{\mathbf{S}})$ where

$\mathbf{S}$ is an affine complex curve with a non-vanishing vector field $\delta_{\mathbf{S}}$

$\mathbf{W}$ is projective over $\mathbf{S}$ with integral geometric fibres of general type

$\mathbf{G}$ is a group scheme of finite type over $\mathbf{S}$ equiped with a vector field $\delta_{\mathbf{G}}$, lifting $\delta_{\mathbf{S}}$, such that the inverse and the multiplication on $\mathbf{G}$ are $\delta_{\mathbf{G}}-$equivariant (call such a $\delta_{\mathbf{G}}$ "group compatible")

$\mathbf{V}$ is a closed subvariety of $\mathbf{G}$, horizontal with respect to $\delta_{\mathbf{G}}$, with a dominant map to $\mathbf{W}$.

A lemma of Hamm says that an "analytic 1-parameter family of complex Lie groups" whose total space is equipped with a "group-compatible" analytic vector field lifting a non-vanishing vector field on the base, is locally analytically trivial. So locally analitically $\mathbf{G} = \mathbf{G}_0 \times \mathbf{S}$, $\delta_{\mathbf{G}} = (0, \delta_{\mathbf{S}})$, where $\mathbf{G}_0$ is some Lie group. (So we have an analytic splitting; note that there is no algebraic splitting in general!) It follows that $\mathbf{V} \to \mathbf{S}$ is locally analytically trivial. Fix $s_0 \in \mathbf{S}$. For any $s \in \mathbf{S}$ consider the analytic map

$$\phi_s : \mathbf{V}_{s_0} \simeq \mathbf{V}_s \to \mathbf{W}_s$$

Its image contains a Zariski open set. The Big Picard Theorem [Kobayashi-Ochiai 75] says that any analytic map from an algebraic variety to a projective variety of general type, whose image contains a complex open set, is in fact algebraic. So our maps ϕ_s are algebraic. So they induce maps from the Albanese variety of (a smooth projective model of) $\mathbf{V}_{s_0}$ into $Alb(\mathbf{W}_s)$. So all $Alb(\mathbf{W}_s)$ are isomorphic to each other and we are done.

b. Sketch of proof of the Finite Absolute Dimension Theorem .

Set $C = Spec\ \mathcal{O}(A^\infty)$. One easily checks that $U := Ker(A^\infty \to A)$ is unipotent. Then one proves that whenever one has an extension

$$0 \to U \to G \to A$$

with A an abelian variety and U unipotent (infinite dimensional) then one also has an exact sequence

$$0 \to \tilde{A} \to G \to \tilde{U} \to 0$$

with $\tilde{A}$ finite dimensional and $\tilde{U}$ unipotent (infinite dimensional). (This is an exercise in the theory of proalgebraic groups.) The one is done by noting that $\tilde{A}$ can be chosen to be a $D-$subgroup and that $A^\sharp$ is the $\delta-$closed subset correspoding to $\tilde{A}$.

c. Proof of Differential Algebraic Lang.

Assume we are in the hypothesis of Differential Algebraic Lang. Then Σ corresponds to an algebraic $D-$subgroup G of A^∞. Let V be an irreducible component of $X^\infty \cap G$ and let $Y \subset X$ be the Zariski closure of the image of V in X. We claim Y is a translate of an abelian subvariety, and we shall be done. Assume it's not. By a result of Ueno there is an abelian subvariety $A_1 \subset A$ such that the image W of Y in A/A_1 is positive dimensional, of general type. By the Theorem on "descent via D-groups" it follows that $Alb(W)$ descends to constants. But A was assumed to have trace zero, a contradiction.

d. Proof of abc for abelian varieties.

Preparation. Let A be an abelian variety over K, with $K/k-$trace zero and let U be an affine open set. The closed set $A \backslash U$ is the support of a very ample effective divisor D on A. Note that D is neither irreducible nor reduced apriori. Embed A into $\mathbf{P}^N_K$ such that D is given by $x_0 = 0$, where $x_0, ..., x_N$ are a basis of $H^0(\mathbf{P}^N_K, \mathcal{O}(1))$. Let $U_j \subset A$ be the open sets defined by $x_j \neq 0$. So $U_0 = U$. We need to show that there is a constant C such that for any point $\eta = (\eta_1, ..., \eta_N) \in U(K)$ and for any place v we have $v(\eta) \geq C$. This actually easily implies the stronger Bounded Multiplicity Theorem as well.

For simplicity we shall assume A is simple.

Constructing the finite set Y. Let A^∞, D^∞ be attached to A and D respectively (the whole construction being made over K, rather than over $\mathcal{F}$). Then D^∞ appears as a closed subscheme of A^∞. Exactly as in the proof of Differential Algebraic Lang, there exists an algebraic $D-$subgroup $G \subset A^\infty$ such that for all $P \in A(K)$ we have $\nabla(P) \in G(K)$. Set $V := D^\infty \cap G$ (scheme theoretic intersection) and let $Y \subset A$ be the Zariski closure of the image of V via the map $A^\infty \to A$; we view Y with its structure of reduced subscheme of A. As in the proof of Differential Algebraic Lang Y is isomorphic, over the algebraic closure of K, with a finite union of translates of abelian subvarieties. Since A was assumed simple, Y is finite.

For simplicity we shall assume the finitely many points of Y are rational over K.

"Uniform discreteness": ϕ_{ij} and γ. It is an easy consequence of the Manin-Chai Theorem of the Kernel that one can find, for each j, a finite family $(\phi_{ij})_{i\in I_j}$, $\phi_{ij} \in \mathcal{J}(Y \cap U_j)$ and there exists a real number $\gamma \geq 0$ such that for any v and any $P \in U_j(K)\backslash Y$, there exists an index $i \in I_j$ such that $v(\phi_{ij}(P)) \leq \gamma$. (Here $\mathcal{J}$ of a subscheme denotes the ideal defining that subscheme.) Identify $\phi_{ij} \in \mathcal{O}(U_j)$ with their pull-backs in $\mathcal{O}(U_j^\infty)$. Then, for a suitable q, we have

$$\phi_{ij}^q \in \mathcal{J}(V \cap U_j^\infty) = \mathcal{J}(D^\infty \cap U_j^\infty) + \mathcal{J}(G \cap U_j^\infty)$$

Clearly

$$\mathcal{J}(D \cap U_j) = \frac{x_0}{x_j} \cdot \mathcal{O}(U_j)$$

So we get

$$\mathcal{O}(D^\infty \cap U_j^\infty) = (\frac{x_0}{x_j}, \delta(\frac{x_0}{x_j}), \delta^2(\frac{x_0}{x_j})...)\mathcal{O}(U_j^n)$$

We may write, for all j and $i \in I_j$:

$$(1) \quad \phi_{ij}^q = \sum_{t=0}^{n} F_{ijt}\left(\frac{x_0}{x_j}, ..., \frac{x_N}{x_j}\right)\delta^t\left(\frac{x_0}{x_j}\right) + g_{ij}$$

with $g_{ij} \in \mathcal{J}(G \cap U_j^\infty) \subset \mathcal{O}(U_j^\infty)$ and F_{ijt} differential polynomials with $K-$coefficients.

Defining β. Clearly we may find a real number $\beta \geq 0$ such that for any $\eta \in K^N$ and any v with $v(\eta) \geq 0$ we have $v(F_{ijt}(\eta)) \geq -\beta$ for all i, j, t.

Fixing P and v. Now we fix a point $P \in U(K) = A(K)\backslash D$ with coordinates $\xi \in K^{N+1}$ and fix a place v. Estimate equation (1) at $\nabla(P) \in A^\infty(K)$. Since $\nabla(P) \in G(K)$ we have $g_{ij}(\nabla(P)) = 0$. So by further taking v we get, for each j, and each $i \in I_j$,:

$$(2) \quad q \cdot v(\phi_{ij}(P)) \geq \min_t\{v\left(F_{ijt}\left(\frac{\xi_0}{\xi_j}, ..., \frac{\xi_N}{\xi_j}\right)\right)\} + \min_t\{v\left(\delta^t\left(\frac{\xi_0}{\xi_j}\right)\right)\}$$

Choosing j. Now for our fixed P and v there exists j such that $\xi_j \neq 0$ (hence $P \in U_j(K)$) and $v(\xi_i/\xi_j) \geq 0$ for all i, i.e. $v(\xi_j) = v(\xi)$. Fix such a j.

Choosing i. For our fixed P, v, j there exists, by "uniform discreteness", an index $i \in I_j$ such that

$$(3) \quad v(\phi_{ij}(P)) \leq \gamma$$

Fix such an index i.

Conclusion. Now, by the choice of β, we have, for all t,

$$(4) \qquad v(F_{ijt}\left(\frac{\xi_0}{\xi_j}, ..., \frac{\xi_N}{\xi_j}\right)) \geq -\beta$$

It is trivial to see that there exists a real number $\theta \geq 0$ such that for all $x \in K$ and all v we have $v(\delta x) \geq v(x) - \theta$. Then putting together (2), (3), and (4), we get

$$q \cdot \gamma \geq q \cdot v(\phi_{ij}(P)) \geq -\beta + (v(\xi_0) - v(\xi_j) - n \cdot \theta)$$

If $\eta = (\xi_1/\xi_0, ..., \xi_N/\xi_0)$ then we get

$$v(\eta) = v(\xi) - v(\xi_0) \geq C := -(\beta + q \cdot \gamma + n \cdot \theta)$$

and we are done.

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