

SUPERSTABLE MANIFOLDS OF INVARIANT CIRCLES

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ABSTRACT

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Let $f : X \dashrightarrow X$ be a dominant meromorphic self-map, where X is a compact, connected complex manifold of dimension $n > 1$. Suppose there is an embedded copy of \mathbb{P}^1 that is invariant under f , with f holomorphic and transversally superattracting with degree a in some neighborhood. Suppose also that f restricted to this line is given by $z \mapsto z^b$, with resulting invariant circle S . We prove that if $a \geq b$, then the local stable manifold $\mathcal{W}_{\text{loc}}^s(S)$ is real analytic. In fact, we state and prove a suitable localized version that can be useful in wider contexts. We then show that the condition $a \geq b$ cannot be relaxed without adding additional hypotheses by presenting two examples with $a < b$ for which $\mathcal{W}_{\text{loc}}^s(S)$ is not real analytic in the neighborhood of any point.

1. INTRODUCTION

Let $f : X \dashrightarrow X$ be a dominant meromorphic self-map of a compact, connected complex manifold X of dimension $n > 1$. Here, the focus is on the situation in which there is $L \subset X$, an embedded copy of \mathbb{P}^1 , with f holomorphic in a neighborhood of L , L is invariant, and $f|L$ is conjugate to $z \mapsto z^b$. We also assume L is transversally superattracting of degree a , that is, the local coordinates of f transverse to L vanishes with order a . Although this is a rather special situation, it has appeared in examples from [1–4].

For such maps, the Julia set of $f|L$ is an invariant circle S , which is a hyperbolic set for f . The local stable manifold $\mathcal{W}_{\text{loc}}^s(S)$ is a real $2n - 1$ dimensional manifold. We will prove:

Theorem A. *If $a \geq b$, then $\mathcal{W}_{\text{loc}}^s(S)$ has real analytic regularity.*

To prove the theorem, we will localize to the situation to a tubular neighborhood of L which is forward invariant under f . Theorem A is a direct consequence of the following:

Theorem A'. *Let N be a complex manifold with $\dim(N) \geq 2$, containing an embedded projective line L . Suppose $f : N \rightarrow N$ a dominant holomorphic map, L is invariant and transversally superattracting with degree a , and $f|L$ is conjugate to $z \mapsto z^b$, having invariant circle S . If $a \geq b$, then $\mathcal{W}_{\text{loc}}^s(S)$ has real analytic regularity.*

In Section 2 we prove Theorem A' by constructing a semi-conjugacy between f and $z \mapsto z^b$ on a forward invariant neighborhood of S .

The proof of Theorem A' is followed by Section 3, where we provide applications to certain specific examples, including those from [2, Section 6.2] and [1]. These examples are followed by Section 4, where an alternative proof of Theorem A for a specific family of maps is provided using holomorphic foliations.

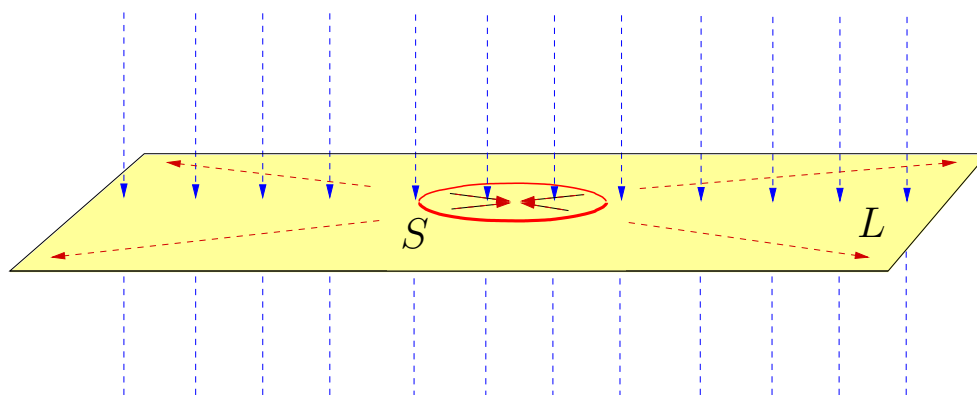


Figure 1.1. Contraction to L and repulsion from S within L

In Section 5, we show that the condition that $a \geq b$ cannot be improved without adding additional hypotheses. We'll consider two maps for which $a < b$ and $\mathcal{W}_{\text{loc}}^s(S)$ is not analytic. One of them is the Migdal-Kadanoff renormalization map R for the Ising model on the Diamond Hierarchical Lattice (DHL) that was studied extensively in [3, 4]. It has $a = 2$ and $b = 4$. The other is a polynomial skew product with $a = 2$ and $b = 3$.

Let us comment a bit more on the map R . For this map, the invariant circle S has the physical context of being related to the bottom of the Lee-Yang cylinder, so it is denoted B . In [4, Lemma 3.2], the authors proved that $\mathcal{W}_{\text{loc}}^s(B)$ is a C^∞ manifold. We prove:

Theorem B. *The stable manifold $\mathcal{W}_{\text{loc}}^s(B)$ is not real analytic at any point.*

Proof of this theorem divides into four main parts. First we construct a co-dimension 1 Böttcher function φ defined in a neighborhood of B under the assumption that $\mathcal{W}_{\text{loc}}^s(B)$ is real analytic. Next we extend the domain of φ to a neighborhood of the set obtained from L by removing the two superattracting fixed points. After that, we develop local properties of R near one of these superattracting fixed points. Lastly, we examine the behavior of φ and R in the extension, from which we derive a contradiction.

This theorem is of physical interest, since $\mathcal{W}_{\text{loc}}^s(B)$ is related to phase transitions of the Ising model on the DHL at low temperatures; see [3, 4]. In §6, we'll explain how Theorem B relates to the limiting distribution of Lee-Yang and Lee-Yang-Fisher zeros at low temperatures.

1.1 Examples Illustrating Hypotheses

Consider a map $f: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ given locally by

$$f(z, w) = (z^2 + wz, w^4 + w^3z).$$

This map satisfies the hypotheses of Theorem A with $L = \{w = 0\}$ and $a = 3 > 2 = b$. Figure 1.1 illustrates several different slices of $\mathcal{W}_{\text{loc}}^s(S)$ in the plane $\{w = c\}$ parallel

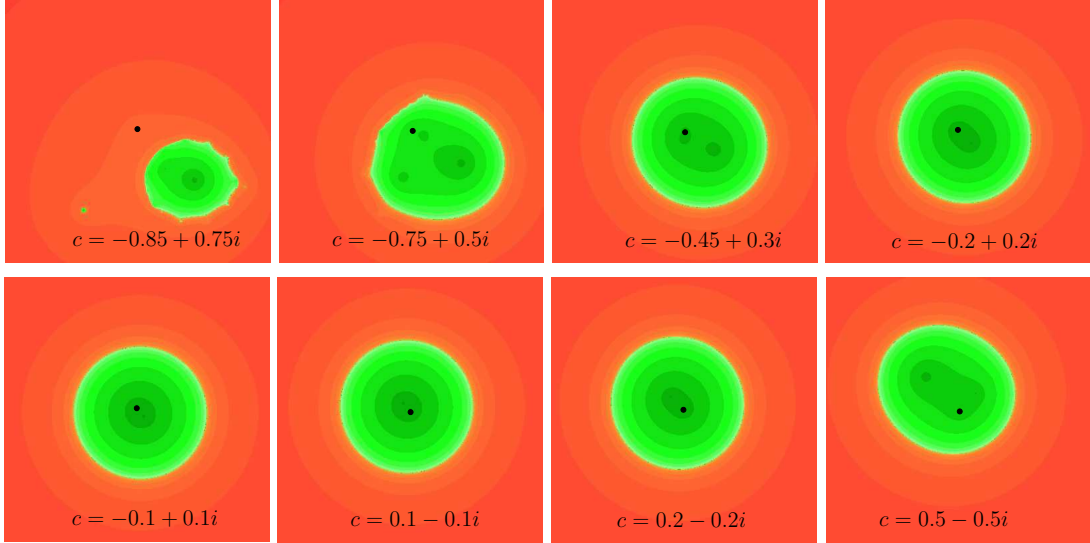


Figure 1.2. $\mathcal{W}^s(S) \cap \{w = c\}$ for f for different c values

to L . In each picture, the visible part of $\mathcal{W}_{\text{loc}}^s(S)$ is the boundary between the two colors, and the gradation of color indicates strength of repulsion from $\mathcal{W}_{\text{loc}}^s(S)$ within $\{w = c\}$.

Now consider $g: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ given locally by

$$g(z, w) = (z^3 + wz^2, w^2).$$

Again, this map satisfies the hypotheses of Theorem A with $L = \{w = 0\}$, with the exception in this case that $a = 2 < 3 = b$. The darker coloring for small $|w|$ in Figure 1.1 illustrates the domination of the repelling direction over the attracting direction.

Both of these illustrate that examination of stable manifolds in this paper is necessarily in a very small neighborhood of S . While the slices of $\mathcal{W}_{\text{loc}}^s(S)$ appear to be at least C^1 for small $|w|$, for $|w|$ even near 1, the behavior of the stable manifold becomes far more complicated.

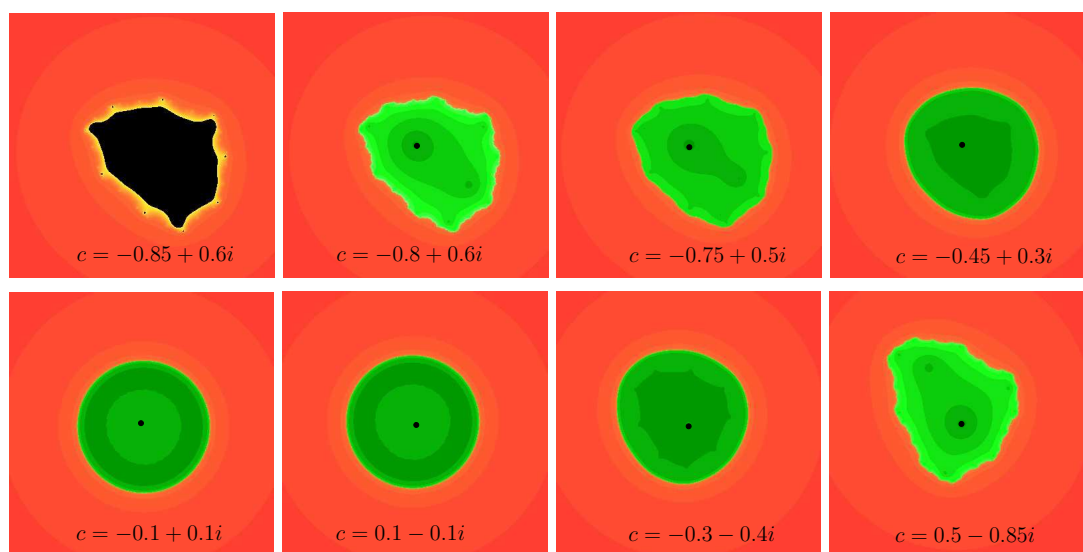


Figure 1.3. $\mathcal{W}^s(S) \cap \{w=c\}$ for g with different c values

1.2 Historical Notes

For a diffeomorphism, the existence and regularity of the local stable manifold for a hyperbolic invariant manifold N has been studied extensively Hirsch-Pugh-Shub in [5]. A strong form of hyperbolicity known as *normal hyperbolicity* is assumed in order to guarantee a C^1 local stable manifold. Specifically, N is called normally hyperbolic for f if the expansion of Df in the unstable direction transverse to N dominates the maximal expansion of Df tangent to N and the contraction of Df in the stable direction transverse to N dominates the maximal contraction of Df tangent to N ; see [5, Theorem 1.1]. For C^r regularity, there is an analogous condition in terms of the r -th power of the maximal expansion/contraction tangent to N .

Although the maps considered in this paper are many-to-one, they also do not fit in the context of [5] since $f|L$ is conformal, forcing that the rates of expansion tangent to S and transverse to S are equal. Thus, S is not normally hyperbolic.

The construction of the semi-conjugacy in the proof of Theorem A' is similar to the proof of the well-known Böttcher's Theorem from one-dimensional complex dynamics [6]; see also [7, Ch. 9].

Theorem 1.2.1 (Böttcher) *If S is a Riemann surface and $f: S \rightarrow S$ is given by $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots$ with $n \geq 2$ and $a_n \neq 0$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to the n th power map $w \mapsto w^n$ throughout some neighborhood of 0. Furthermore, ϕ is unique up to multiplication by an $(n-1)$ st root of unity.*

In fact, many of the techniques used in this paper are similar to this classical theorem in spirit. While Böttcher's Theorem refers to a holomorphic change of coordinate (often called a Böttcher coordinate) defined in the neighborhood of a superattracting fixed point, the function we construct here is neither a coordinate, nor is it defined in a full neighborhood of a superattracting fixed point. However, by analogy, we call it a "co-dimension 1 Böttcher function."

Those interested in the mathematical legacy of Böttcher should see [8]. We will now briefly describe variants of Böttcher's Theorem in higher dimensions. It was shown by Hubbard and Papadopol in [9] that a Böttcher coordinate in higher dimension cannot exist in general. With additional hypotheses, their existence has been proved in [10, Theorem 3.2] and [11]. A more detailed criterion for existence of a Böttcher coordinate is presented in [12]. The related problem of conjugating a polynomial endomorphism to its highest degree terms in a neighborhood of the hyperplane at infinity is studied in [9, Theorem 9.3], [13, Theorem 7.4], [14], and [15, Theorem 1]. These authors prove that such a conjugacy exists on the stable set of the Julia set at infinity, so long as it satisfies suitable hyperbolicity. More recent studies of superattracting behavior appear in [16, 17].

2. PROOF OF THEOREM A'

The \mathbb{C}^{n-1} bundle over \mathbb{P}^1 can be described by two systems of locally trivializing coordinates $(z, \mathbf{w}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and $(\zeta, \boldsymbol{\omega}) \in \mathbb{C} \times \mathbb{C}^{n-1}$. For $z \neq 0$, they are related by $\zeta = 1/z$ and $\boldsymbol{\omega} = A_z \mathbf{w}$, with $A_z : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ a linear isomorphism depending holomorphically on z . Let us choose these trivializations so that the dynamics on the zero section is $z \mapsto z^b$.

We will make use of standard multi-index notation. Given $\mathbf{c} \in \mathbb{Z}_+^{n-1}$ and $\mathbf{w} \in \mathbb{C}^{n-1}$, $\mathbf{w}^{\mathbf{c}} = w_1^{c_1} w_2^{c_2} \cdots w_{n-1}^{c_{n-1}}$ and $|\mathbf{c}| = c_1 + \cdots + c_{n-1}$. We will always use the standard Hermitian norm $|\mathbf{w}| = (|w_1|^2 + \cdots + |w_{n-1}|^2)^{1/2}$ on \mathbb{C}^{n-1} .

Lemma 2.0.2 *There are holomorphic functions \mathbf{g}_1 and $\mathbf{g}_{\mathbf{c}}$ for each $|\mathbf{c}| = a$ such that in the (z, \mathbf{w}) coordinates*

$$f(z, \mathbf{w}) = \left(z^b + \mathbf{w} \cdot \mathbf{g}_1(z, \mathbf{w}), \sum_{|\mathbf{c}|=a} \mathbf{w}^{\mathbf{c}} \mathbf{g}_{\mathbf{c}}(z, \mathbf{w}) \right).$$

Similarly, there are holomorphic functions \mathbf{h}_1 and $\mathbf{h}_{\mathbf{c}}$ for each $|\mathbf{c}| = a$ such that in the $(\zeta, \boldsymbol{\omega})$ coordinates

$$f(\zeta, \boldsymbol{\omega}) = \left(\zeta^b + \boldsymbol{\omega} \cdot \mathbf{h}_1(\zeta, \boldsymbol{\omega}), \sum_{|\mathbf{c}|=a} \boldsymbol{\omega}^{\mathbf{c}} \mathbf{h}_{\mathbf{c}}(\zeta, \boldsymbol{\omega}) \right).$$

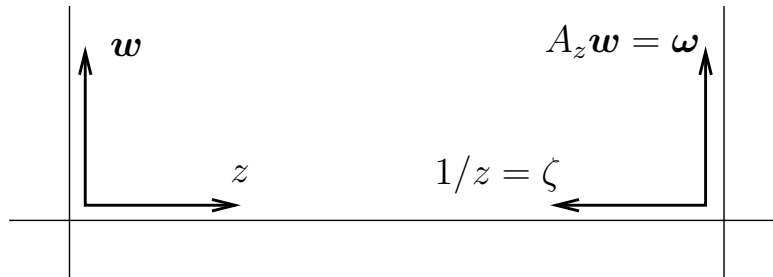


Figure 2.1. Local coordinates centered on the two fixed points in L

Proof The proof is the same in both coordinate systems, so we'll work in the (z, \mathbf{w}) system. Since $f|L$ is the map $z \mapsto z^b$, the first coordinate of f minus z^b vanishes on L . Since L is given by $\mathbf{w} = \mathbf{0}$, we have that the first coordinate of f is $z^b + \mathbf{w} \cdot \mathbf{g}_1(z, \mathbf{w})$ for some holomorphic function \mathbf{g}_1 . Meanwhile, the expression for the second coordinate follows from the fact that L is transversally superattracting of degree a . ■

2.1 Hyperbolic theory

We'll now verify that the local stable manifold $\mathcal{W}_{\text{loc}}^s(S)$ is a $2n-1$ real-dimensional topological manifold that is foliated by local stable manifolds of each point of S .

The hyperbolic theory for endomorphisms is somewhat less standard than for diffeomorphisms. Suitable references from the context of complex dynamics include [13, 18, 19]. For consistency, we will use definitions and results from [13, Appendix B]. Let us consider the natural extension

$$\hat{S} := \{(x_i)_{i \leq 0} : x_i \in S \text{ and } f(x_i) = x_{i+1}\}.$$

We'll denote such histories by $\hat{x} = (x_i)_{i \leq 0} \in \hat{S}$. Notice that the action of f naturally lifts to an action $\hat{f} : \hat{S} \rightarrow \hat{S}$.

Lemma 2.1.1 *S is a hyperbolic set for the map f .*

Proof Note that for $x \in S$, we have

$$Df_x = \begin{bmatrix} bz^{b-1} & \frac{\partial}{\partial \mathbf{w}} \mathbf{g}_1(z, \mathbf{0}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus, we have $E^s(x) = \ker(Df)$ and $E^u(\hat{x}) \subset L$, so $T_x \mathbb{C}^n = E^s(x) \oplus E^u(\hat{x})$. Invariance of $E^s(x)$ follows from the fact any point in the kernel is collapsed to $(0, \mathbf{0})$ under Df , and invariance of $E^u(\hat{x})$ follows from the invariance of L . Also, for any $v^s \in E^s(x)$ and $v^u \in E^u(\hat{x})$ with $n \geq 0$,

$$\|Df_x^n v^s\| = 0 \leq C\lambda^n \|v^s\| \text{ and } \|Df_x^n v^u\| \leq C\lambda^{-n} \|v^u\|,$$

for $C = 1$ and $\lambda = 1/2$. Thus, we have that S is hyperbolic. ■

Therefore, by the stable manifold theorem (see, for example, [20, Theorem 5.2]) each point $x \in S$ will have local stable manifold $\mathcal{W}_{\text{loc}}^s(x)$ that is a complex $n - 1$ ball holomorphically embedded into N and each prehistory \hat{x} will have a local unstable manifold $\mathcal{W}_{\text{loc}}^u(\hat{x})$, which is a holomorphic disc. They depend continuously on x and \hat{x} . (In this case, the unstable manifolds all lie in L .)

Existence of such stable laminations has also been proved in the holomorphic context by Ushiki [21]. It can be proved in the following simple way as well, which is a direct generalization of what was done in [1, Proposition 4.2] and [3, Proposition 9.2].

By the stable manifold theorem for a point (see, for example, [22, Section 2.6] or [11], which hold even if Df has an eigenvalue of 0), there exists a local stable manifold, $\mathcal{W}_{\text{loc}}^s((1, \mathbf{0}))$, which is the graph of a holomorphic function $z = \eta_1(\mathbf{w})$ defined on some $(n - 1)$ -dimensional open ball, Λ , in the \mathbf{w} axis. Let $\Sigma \subset S$ to be the set of iterated preimages of $(1, \mathbf{0})$. Using a suitable invariant cone field and a well-chosen neighborhood of S , one can take iterated preimages of $\mathcal{W}_{\text{loc}}^s((1, \mathbf{0}))$ so that the preimage through each $x \in \Sigma$ is expressed as the graph of a holomorphic function $\eta_x(\mathbf{w})$ defined on Λ , making Λ smaller if necessary. In this way, we can construct local stable manifolds over Σ , which is dense in S . The function $\eta: \Lambda \times \Sigma \rightarrow \mathbb{C}$ given by $\eta(\mathbf{w}, x) = \eta_x(\mathbf{w})$ defines a holomorphic motion of $\Sigma \subset \mathbb{C}$, parameterized by $\mathbf{w} \in \Lambda \subset \mathbb{C}^{n-1}$. We may use the λ -lemma [23, 24] to extend η continuously to a holomorphic motion of $\bar{\Sigma} = S$, obtaining stable manifolds for every point of S .

Definition 2.1.1 *A hyperbolic set $\hat{\Lambda}$ has a local product structure, if $\delta > 0$ can be chosen small enough so that for any $p \in \Lambda$ and $\hat{q} \in \hat{\Lambda}$, either $\mathcal{W}_\delta^s(p) \cap \mathcal{W}_\delta^u(\hat{q})$ is empty or it is a single point $x \in \Lambda$ so the unique history \hat{x} of x satisfying $x_j \in \mathcal{W}_\delta^u(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ is completely contained in $\hat{\Lambda}$.*

Lemma 2.1.2 *S has local product structure for the map f .*

Proof By Lemma 2.1.1, S is hyperbolic. Recall that for any $\hat{q} \in \hat{S}$, we have that $\mathcal{W}_\delta^u(\hat{q}) = \mathbb{D}_\delta(q_0) \subset L$, which is the disc of radius $\delta > 0$ centered at the point q

contained in L . Since $\mathcal{W}_\delta^u(\hat{q})$ depends only on q_0 , existence of a local product structure for \hat{S} is very simple.

By the Stable Manifold Theorem, we may choose $\delta > 0$ small enough so that for any $p \in S$, we have $\mathcal{W}_\delta^s(p) \cap L = \{p\}$. Thus, for any two points $p, q \in S$, the intersection $\mathcal{W}_\delta^s(p) \cap \mathcal{W}_\delta^u(\hat{q}) = \{p\}$, with $p \in S$. Moreover, p has a unique prehistory $\hat{p} = (p_i)_{i \leq 0}$ with $p_j \in \mathcal{W}_\delta^u(\hat{f}^j(\hat{q}))$ for all $j \leq 0$, and it is completely contained in \hat{S} as well. \blacksquare

Given a neighborhood Ω of S , let

$$\mathcal{W}_{\text{loc}}^s(S) := \{x \in N : f^n x \in \Omega \text{ and } f^n x \rightarrow S \text{ as } n \rightarrow \infty\} \quad (2.1)$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}_{\text{loc}}^s(S)$ means that it holds for any sufficiently small neighborhood of S).

Since S has a local product structure $\mathcal{W}_{\text{loc}}^s(S)$ is the union of the local stable manifolds $\mathcal{W}_{\text{loc}}^s(x)$ of points $x \in \mathcal{B}$; see [13, Proposition B.6]. The local stable manifolds of points are pairwise disjoint and depend continuously on the base point, therefore we have:

Corollary 2.1.3 $\mathcal{W}_{\text{loc}}^s(S)$ is a topological manifold of real dimension $2n - 1$.

2.2 Co-dimension 1 Böttcher function

Let $(z_n, \mathbf{w}_n) := f^n(z, \mathbf{w})$. Motivated by Böttcher's theorem [6], [7, p. 86], we consider a sequence of functions

$$\varphi_n(z, \mathbf{w}) = z_n^{1/b^n}.$$

We will show that the φ_n converge uniformly on compact subsets of some forward invariant neighborhood Ω of S to a holomorphic function φ that semi-conjugates f to $z \mapsto z^b$:

$$\varphi(f(z, \mathbf{w})) = \varphi(z, \mathbf{w})^b. \quad (2.2)$$

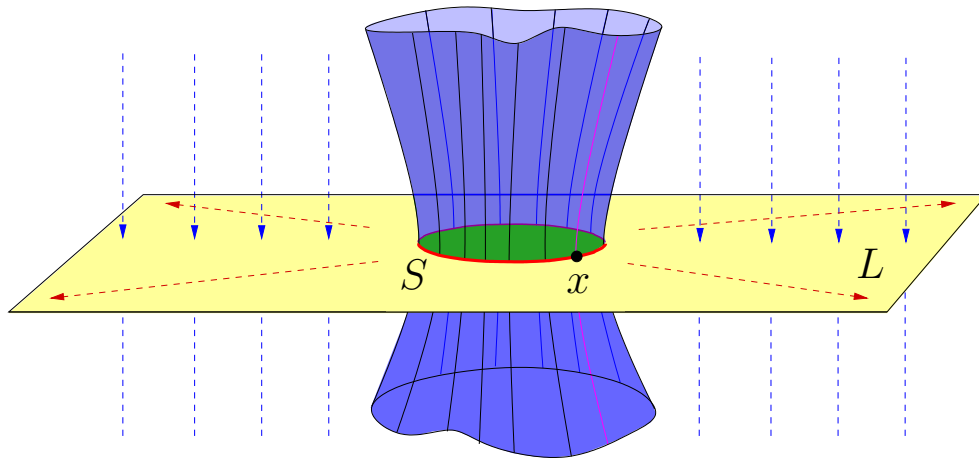


Figure 2.2. $W_{\text{loc}}^s(S)$, contraction to L , and repulsion from S within L

To make sense of the b^n -th roots and the limit, we'll rewrite each φ_n as telescoping product:

$$\varphi = \lim_{n \rightarrow \infty} \varphi_n = z_0 \cdot \frac{z_1^{1/b}}{z_0} \cdot \frac{z_2^{1/b^2}}{z_1^{1/b}} \cdot \frac{z_3^{1/b^3}}{z_2^{1/b^2}} \cdots = z_0 \prod_{n=0}^{\infty} \left(\frac{z_{n+1}}{z_n^b} \right)^{\frac{1}{b^{n+1}}}, \quad (2.3)$$

where it follows from Lemma 2.0.2 that

$$\frac{z_{n+1}}{z_n^b} = \frac{z_n^b + \mathbf{w}_n \cdot \mathbf{g}_1(z_n, \mathbf{w}_n)}{z_n^b} = 1 + \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n). \quad (2.4)$$

In the $(\zeta, \boldsymbol{\omega})$ coordinates we have:

$$\frac{z_{n+1}}{z_n^b} = \frac{\zeta_n^b}{\zeta_{n+1}} = \frac{1}{1 + \frac{\boldsymbol{\omega}_n}{\zeta_n^b} \cdot \mathbf{h}_1(\zeta_n, \boldsymbol{\omega}_n)}. \quad (2.5)$$

When working in $\mathcal{W}^s(\eta_1)$ we'll use expression (2.4), when working in $\mathcal{W}^s(\eta_2)$ we'll use expression (2.5), and when working on $\mathcal{W}_{\text{loc}}^s(S)$, we'll use either.

We'll construct a forward invariant neighborhood Ω of S so that if $(z, \mathbf{w}) \in \Omega \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$, then

$$\left| \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n) \right| < \frac{1}{2}, \quad (2.6)$$

and if $(\zeta, \boldsymbol{\omega}) \in \Omega \cap (\mathcal{W}^s(\eta_2) \cup \mathcal{W}_{\text{loc}}^s(S))$, then

$$\left| \frac{\boldsymbol{\omega}_n}{\zeta_n^b} \cdot \mathbf{h}_1(\zeta_n, \boldsymbol{\omega}_n) \right| < \frac{1}{2}. \quad (2.7)$$

Then, for points in Ω , the b^n -th root is defined by taking the branch cut along the negative real axis. Moreover, this condition will also imply convergence of the infinite product (2.3) on Ω , since the corresponding sum of logarithms converges:

$$\sum_{n=1}^{\infty} \log \left| \frac{z_{n+1}}{z_n^b} \right|^{\frac{1}{b^{n+1}}} \leq \sum_{n=1}^{\infty} \frac{1}{b^{n+1}} \log 2.$$

To construct Ω , first note that for any $K_1 > 0$ sufficiently small, $\{|\mathbf{w}| \leq K_1\} \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$ is a compact subset of \mathbb{C}^n . Since \mathbf{g}_1 is holomorphic on \mathbb{C}^n , there is a bound $|\mathbf{g}_1(z, \mathbf{w})| \leq K_2$ on any such compact set. A similar bound holds in the other coordinate system. Therefore, it suffices to show:

Lemma 2.2.1 *Given any $K > 0$, there exists a forward invariant neighborhood of S in which*

$$\frac{|\mathbf{w}|}{|z|^b} < K \quad \text{and} \quad \frac{|\boldsymbol{\omega}|}{|\zeta|^b} < K. \quad (2.8)$$

Proof of this lemma relies on point blow-ups, so let us provide a brief description of this technique using the definitions from [25]. Let M be a complex manifold of dimension n , and let $z = (z_1, \dots, z_n)$ holomorphic coordinates in an open set $U \subset M$ centered around the point $p \in M$. The *blow-up* \tilde{M} of M at p is the complex manifold obtained by adjoining to $M \setminus \{p\}$ the manifold

$$\tilde{U} = \{(z, l) : z \in U\} \subset U \times \mathbb{P}^{n-1}$$

via the isomorphism

$$\tilde{U} \setminus (z = 0) \cong U \setminus \{p\}$$

given by $(z, l) \mapsto z$. One may be tempted to think of this simply as a holomorphic change of coordinates, and while that is true, what is happening additionally is that the point p is being replaced by a projective hypersurface. There is a natural projection map $\pi: \tilde{M} \rightarrow M$ extending the identity on $M \setminus \{p\}$. By construction, $E = \pi^{-1}(p)$ is isomorphic to $\mathbb{C}P^{n-1}$ and is called the *exceptional divisor* of the blow-up $\tilde{M} \rightarrow M$.

Proof [Proof of Lemma 2.2.1] We will take an inductive sequence of b point blow-ups at each of the two fixed points η_1 and η_2 . Using the forms of f given by Lemma 2.0.2, the calculation will be the same at each of these two points, so we'll focus on η_1 , which is given by $(z, \mathbf{w}) = (0, \mathbf{0})$.

We first do a point blow-up at η_1 , producing an exceptional divisor $E_{\eta_1,1}$. Let \tilde{L}_1 be the proper transform of L . We then blow-up the point intersection point between $E_{\eta_1,1}$ and \tilde{L}_1 , producing a new exceptional divisor $E_{\eta_1,2}$ and proper transform \tilde{L}_2 . We inductively do this $b-2$ additional times, each time blowing up the intersection point between the previous exceptional divisor and proper transform of L .

Consider the system of coordinates $z, \boldsymbol{\lambda} = \frac{\boldsymbol{w}}{z^b}$ centered at the intersection point of $E_{\eta_1, b}$ with \tilde{L}_b . Let us denote $(z', \boldsymbol{\lambda}') = \tilde{f}(z, \boldsymbol{\lambda})$, where \tilde{f} is the extension of f to the final blow-up. We have

$$\begin{aligned} z' &= z^b + z^b \boldsymbol{\lambda} \cdot \boldsymbol{g}(z, z^b \boldsymbol{\lambda}) \\ \boldsymbol{\lambda}' &= \frac{\boldsymbol{w}'}{(z')^b} = \frac{\sum_{|\mathbf{c}|=a} (z^b \boldsymbol{\lambda})^{\mathbf{c}} \boldsymbol{g}_{\mathbf{c}}(z, z^b \boldsymbol{\lambda})}{(z^b + z^b \boldsymbol{\lambda} \cdot \boldsymbol{g}(z, z^b \boldsymbol{\lambda}))^b} = \frac{z^{b(a-b)} \sum_{|\mathbf{c}|=a} \boldsymbol{\lambda}^{\mathbf{c}} \boldsymbol{g}_{\mathbf{c}}(z, z^b \boldsymbol{\lambda})}{(1 + \boldsymbol{\lambda} \cdot \boldsymbol{g}(z, z^b \boldsymbol{\lambda}))^b}. \end{aligned}$$

Notice that this extension \tilde{f} is holomorphic in a neighborhood of $(z, \boldsymbol{\lambda}) = (0, \mathbf{0})$ and that this point is superattracting for \tilde{f} .

Therefore, for any $\varepsilon_1 > 0$ and $K \geq \delta_1 > 0$, sufficiently small, $\tilde{U}_1 := \{|z| < \varepsilon_1, |\boldsymbol{\lambda}| < \delta_1\}$ will be forward invariant under \tilde{f} . Hence,

$$U_1(\varepsilon_1, \delta_1) := \pi \left(\tilde{U}_1(\varepsilon_1, \delta_1) \right) = \left\{ |z| < \varepsilon_1, \frac{|\boldsymbol{w}|}{|z|^b} < \delta_1 \right\}$$

will be a forward invariant set for f .

As stated before, the same calculation can be done at η_2 , with analogous results. In particular, for any $\varepsilon_2 > 0$ and $K \geq \delta_2 > 0$ sufficiently small we will have a forward invariant set for f of the form

$$U_2(\varepsilon_2, \delta_2) = \left\{ |\zeta| < \varepsilon_2, \frac{|\boldsymbol{w}|}{|\zeta|^b} < \delta_2 \right\}.$$

Let $V \subset N$ be a forward invariant tubular neighborhood of L and let

$$V(\varepsilon_1, \varepsilon_2) = V \setminus (\{|z| < \varepsilon_1\} \cup \{|\zeta| < \varepsilon_2\}).$$

Note that if V sufficiently small, then all points of $V(\varepsilon_1, \varepsilon_2)$ satisfy (2.8). We will show that V can be made even smaller, if necessary, in order to make

$$\Omega := V(\varepsilon_1, \varepsilon_2) \cup U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$$

forward invariant.

Since $U_1(\varepsilon_1, \delta_1)$ and $U_2(\varepsilon_2, \delta_2)$ are forward invariant, we need only check that if $x \in V(\varepsilon_1, \varepsilon_2)$ and $f(x) \notin V(\varepsilon_1, \varepsilon_2)$, then $f(x) \in U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$. Let us focus on $x \in \mathcal{W}^s(\eta_1)$, since the proof will be the same for $x \in \mathcal{W}^s(\eta_2)$.

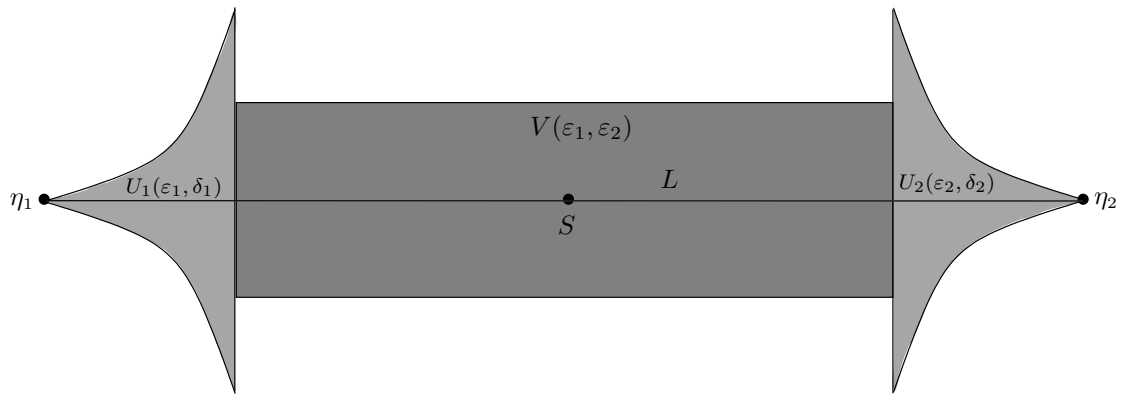


Figure 2.3. The forward invariant neighborhood Ω

Let $x = (z, \mathbf{w}) \in V(\varepsilon_1, \varepsilon_2) \cap \mathcal{W}^s(\eta_1)$ and let $(z_1, \mathbf{w}_1) = f(z, \mathbf{w})$. Since $(z, \mathbf{w}) \in V(\varepsilon_1, \varepsilon_2)$, $|\mathbf{w}|/|z|^b < K$, so that (2.6) and (2.4) imply that the $|z_1| \geq |z|^b/2 \geq \varepsilon_1^b/2$. Thus, we need only choose the (forward invariant) tubular neighborhood V sufficiently small so that

$$V \cap \left\{ \frac{\varepsilon_1^b}{2} \leq |z| \leq \varepsilon_1 \right\} \subset U_1(\varepsilon_1, \delta_1).$$

Doing the same thing near η_2 , we construct a forward invariant neighborhood Ω satisfying (2.8). ■

2.3 Completing the proof of Theorem A'

Using the invariance (2.2), for any $(z, \mathbf{w}) \in \mathcal{W}_{\text{loc}}^s(S)$ we have $|\varphi(z, \mathbf{w})| = 1$ so that $\psi := \log |\varphi|$ will be a real analytic function that vanishes on $\mathcal{W}_{\text{loc}}^s(S)$. Notice on that L , we have $\varphi(z, \mathbf{0}) = z$ and hence $\psi(z, \mathbf{0}) = \log |z|$. Since the derivative $D\psi$ is non-zero on S , we have that $\{\psi = 0\}$ is a real analytic $2n - 1$ real-dimensional manifold in some neighborhood of S .

By Corollary 2.1.3, $\mathcal{W}^s(S) \subset \{\psi = 0\}$ is also a real $2n - 1$ dimensional manifold. Thus, by invariance of domain, $\mathcal{W}^s(S) = \{\psi = 0\}$ in this neighborhood.

3. EXAMPLES ILLUSTRATING THEOREM A

3.1 Regular Polynomial Endomorphisms in Two Dimensions.

Suppose $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a degree d regular polynomial endomorphism. Then f has the form $f(x, y) = (p(x, y), q(x, y))$ for polynomials p and q . Moreover, if $p_d(x, y)$ and $q_d(x, y)$ are the degree d homogeneous terms of p and q , then in homogeneous coordinates in the line at infinity, $L_\infty = \{Z = 0\}$, f has the form

$$f|_{Z=0} [X : Y : Z] = [p_d(X, Y) : q_d(X, Y) : 0]. \quad (3.1)$$

Since f is regular, p_d and q_d have no common zeros, so there is no indeterminacy on L_∞ . The coordinate on L_∞ is $z = Y/X$, so if we assume $f|_{L_\infty}$ is conjugate to $z \mapsto z^d$, then there are coordinates such that $p_d(X, Y) = X^d$ and $q_d(X, Y) = Y^d$. Then

$$f_\infty(z) = \frac{p_d(1, z)}{q_d(1, z)} = z^d, \quad (3.2)$$

so J_∞ , the Julia set on L_∞ , is a geometric circle. Thus, this situation satisfies the hypotheses of Theorem A with $a = b = d$, so we have the following:

Corollary 3.1.1 *If f is a regular polynomial endomorphism of \mathbb{C}^2 for which $f|_{L_\infty}$ is conjugate to $z \mapsto z^d$, then $\mathcal{W}_{\text{loc}}^s(J_\infty)$ has real analytic regularity.*

Real analyticity of the stable manifold considered in [2, Section 6.2] is a direct application of Corollary 3.1.1.

3.2 Degenerate Newton Mappings

Newton mappings used to find the common roots of $P(x, y) = x(1 - x)$ and $Q(x, y) = y^2 + Bxy - y$ were considered dynamically in [1]. They have the form

$$N(x, y) = \left(\frac{x^2}{2x - 1}, \frac{y(Bx^2 + 2xy - Bx - y)}{(2x - 1)(Bx + 2y - 1)} \right). \quad (3.3)$$

We will consider their extension as rational maps of $\mathbb{P}^1 \times \mathbb{P}^1$. They are skew products with the first coordinate having superattracting fixed points of degree 2 at $x = 0$ and $x = 1$, so the vertical lines $\{x = 0\} \times \mathbb{P}^1$ and $\{x = 1\} \times \mathbb{P}^1$ are transversally superattracting for N with the same degree. Using the formula, one can check that N has no indeterminate points in some neighborhood of these two lines.

Restricted to $\{x = 0\} \times \mathbb{P}^1$, N is the one-dimensional Newton map for the quadratic polynomial with roots at $y = 0$ and $y = 1$. It is therefore conjugate to $z \mapsto z^2$, having an invariant circle S_0 corresponding to the points of equal distance from $y = 0$ and $y = 1$ in \mathbb{P}^1 . (S_0 is the closure of $\text{Im}(y) = \frac{1}{2}$ in \mathbb{P}^1 .)

Similarly, the restriction of N to $\{x = 1\} \times \mathbb{P}^1$ is the one-dimensional Newton map for the quadratic polynomial with roots at $y = 0$ and $y = 1 - B$. Thus, it is conjugate to $z \mapsto z^2$, with an invariant circle S_1 corresponding to the points of equal distance from $y = 0$ and $y = 1 - B$ within \mathbb{P}^1 .

Both of the lines $\{0\} \times \mathbb{P}^1$ and $\{1\} \times \mathbb{P}^1$ is transversally superattracting with degree 2, with the restriction of N to each of them conjugate to $z \mapsto z^2$. Therefore, it follows immediately from Theorem A that the local stable manifolds $\mathcal{W}_{\text{loc}}^s(S_0)$ and $\mathcal{W}_{\text{loc}}^s(S_1)$ are real analytic. This was proven previously in [1] using more specific details of the mapping.

3.3 Example with indeterminacy

Consider the polynomial mapping $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$g(x, y) = (x^2 + y(1 + xy), y^3(1 + xy)). \quad (3.4)$$

Within \mathbb{C}^2 , the line $L := \{y = 0\}$ is invariant and transversally superattracting with degree 3 and $g|_L$ is given by $x \mapsto x^2$. Let $S := \{|x| = 1, y = 0\}$ be the invariant circle. Although there is the needed domination between the degrees ($3 > 2$), to apply Theorem A we need to check how g extends to a neighborhood of infinity on L . The extension of g to \mathbb{P}^2 is given in homogeneous coordinates by

$$g[X : Y : Z] = [X^2Z^3 + YZ^2(Z^2 + XY) : Y^3(Z^2 + XY) : Z^5].$$

There is a point of indeterminacy for g at $[1 : 0 : 0]$ on the projective line $Y = 0$, which we'll also denote by L . Therefore, Theorem A does not immediately apply.

Let us perform two blowups. We first blow-up the point $[1 : 0 : 0]$ and we then blow-up the point where the proper transform of L intersects the exceptional divisor over $[1 : 0 : 0]$. We'll denote the space obtained after doing these two blow-ups by $\widetilde{\mathbb{P}^2}$, the projection by $\pi : \widetilde{\mathbb{P}^2} \rightarrow \mathbb{P}^2$, the proper transform of L after these two blow-ups by \widetilde{L} , the invariant circle within \widetilde{L} by \widetilde{S} , and the lift of g to the blown-up space by $\widetilde{g} : \widetilde{\mathbb{P}^2} \rightarrow \widetilde{\mathbb{P}^2}$.

A neighborhood of \widetilde{L} can be described by two systems of coordinates (x, y) and (ζ, τ) , where $x = X/Z, y = Y/Z$ are the original affine coordinates on \mathbb{C}^2 and $\zeta = Z/X, \tau = XY/Z^2$. In the first system of coordinates, \widetilde{g} is given by (3.4). In the second system of coordinates, \widetilde{g} is given by

$$\widetilde{g}(\zeta, \tau) = \left(\frac{\zeta^2}{1 + \tau \zeta^3(1 + \tau)}, \tau^3 \zeta (1 + \tau) (1 + \tau \zeta^3 + \tau^2 \zeta^3) \right).$$

In the second system of coordinates, \widetilde{L} is given by $\tau = 0$, so we see that \widetilde{g} is holomorphic in a neighborhood of \widetilde{L} . Moreover, \widetilde{L} is invariant and transversally superattracting with degree 3 and $\widetilde{g}|_{\widetilde{L}}$ still given by $x \mapsto x^2$. Therefore, Theorem A applies to give that the local stable manifold $\mathcal{W}_{\text{loc}}^s(\widetilde{S})$ for \widetilde{S} under \widetilde{g} is real analytic.

Notice that \widetilde{g} and g are birationally conjugate by means of π . Moreover, restricted to small neighborhoods of \widetilde{S} and S , this birational conjugacy becomes an honest holomorphic conjugacy. Since the local stable manifolds $\mathcal{W}_{\text{loc}}^s(\widetilde{S})$ and $\mathcal{W}_{\text{loc}}^s(S)$ are defined in terms of the action of iterates of \widetilde{g} and g , respectively, on these small neighborhoods, we conclude that $\mathcal{W}_{\text{loc}}^s(S)$ is also real analytic.

This third example illustrates two important considerations about Theorem A. First, it illustrates that one sometimes needs to do some blow-ups in order to obtain a map without indeterminacy in a neighborhood of L .

Second, it illustrates the reason why we need to consider arbitrary \mathbb{C}^{n-1} bundles over L , since blowing up points of L will change its normal bundle. (In this example,

the normal bundle of L is the hyperplane bundle, while the normal bundle of \tilde{L} is the tautological bundle.)

4. EXAMPLE OF THEOREM A PROVED USING THE FOLIATION

Consider the family of skew product maps $f_\lambda: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$f_\lambda(z, w) = (z^2 + \lambda w, w^2),$$

for $\lambda \in \mathbb{C}$. The lines $L_0 := \{w = 0\}$ and $L_1 := \{z = 0\}$ are both invariant. Moreover, L_0 is transversally super-attracting, and $f_\lambda|_{L_0}$ is the map $z \mapsto z^2$. Again, $S := \{|z| = 1, w = 0\} \subset L_0$ is an invariant circle contained in L_0 .

Thus, this specific example satisfies the hypotheses of Theorem A with $a = b = 2$, so

Corollary 4.0.1 *For any $\lambda \in \mathbb{C}$, the map f_λ generates the stable manifold $\mathcal{W}_{\text{loc}}^s(S)$ with real analytic regularity.*

In this situation, however, we present a proof of this result using a different technique. The corollary will be proved by constructing a backward invariant holomorphic foliation in a neighborhood of S . This foliation will have the property that each leaf intersects L_0 in a unique point, and the projection to L_0 along leaves is a holomorphic function. Since $\mathcal{W}_{\text{loc}}^s(S)$ is the preimage under the projection of the real analytic set S , $\mathcal{W}_{\text{loc}}^s(S)$ must also be real analytic.

4.1 Backward Invariant Neighborhoods and Cone Fields

To iteratively pull back a holomorphic foliation of L_0 in a neighborhood of S , we will need a backward invariant set that avoids critical values of f_λ (besides L_0) in order to maintain a proper holomorphic foliation. It follows from $\det(Df) = 4zw$ that L_0 and L_1 are the only critical curves, so let $B := \{|w| \geq c|z|^2\}$ for some $c > 0$.

Lemma 4.1.1 *There is a $c > 0$ such that B is backward invariant.*

Proof Suppose $f_\lambda(z, w) = (z', w') \in B$, so we have

$$c \leq \frac{|w'|}{|z'|^2} = \frac{|w|^2}{|z^2 - aw|^2} = \frac{|w/z^2|^2}{|1 - a(w/z^2)|^2}.$$

Let $x = w/z^2$, so

$$\sqrt{c} \leq \frac{|x|}{|1 + ax|} \leq \frac{|x|}{|1 - |ax||},$$

which yields two cases.

Case one: If $1 > |ax|$, then $\sqrt{c} \leq |x|/(1 - |ax|)$, which implies $\sqrt{c} \leq \sqrt{|x| + c|ax|}$,

or

$$\frac{\sqrt{c}}{1 + \sqrt{c}|a|} \leq |x|.$$

Then

$$c \leq \frac{\sqrt{c}}{1 + \sqrt{c}|a|} \leq \left| \frac{w}{z^2} \right|$$

precisely when $c|a| + \sqrt{c} - 1 \leq 0$, or

$$c \leq \left(\frac{-1 + \sqrt{1 + 4|a|}}{2|a|} \right)^2. \quad (4.1)$$

Case two: If $1 < |ax|$, then $\sqrt{c} \leq |x|/(|ax| - 1)$, which implies $(\sqrt{c}|a| - 1)|x| = \sqrt{c}|ax| - |x| \leq \sqrt{c}$. Then if $c \leq 1/|a|^2$,

$$\sqrt{c} \leq \frac{\sqrt{c}}{1 - \sqrt{c}|a|} \leq |x|.$$

in which case, we again have $c \leq |w/z^2|$.

It follows that for

$$c \leq \min \left\{ \frac{1}{|a|^2}, \left(\frac{-1 + \sqrt{1 + 4|a|}}{2|a|} \right)^2 \right\} \quad (4.2)$$

that B is backward invariant. ■

The behavior of f_λ near infinity is also relevant, so extend f_λ to a map $F_\lambda: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ by

$$F_\lambda[Z : W : T] = [Z^2 +_\lambda WT : W^2 : T^2],$$

and use local coordinates $\omega = W/Z$ and $\tau = T/Z$, so

$$\omega' = \frac{\omega^2}{1 + a\omega\tau} \quad \text{and} \quad \tau' = \frac{\tau^2}{1 + a\omega\tau}$$

Then we have $|DF_\lambda| = \frac{4\omega\tau}{(1+a\omega\tau)^3} = 0$ only on $\{\omega = 0\}$ and $\{\tau = 0\}$, and both of these lines are totally invariant. Let $B' := \{|\tau| \geq c|\omega|^2\}$, where c is the constant from Lemma 4.1.1. Observe that in these coordinates,

$$c \leq \frac{|\tau'|}{|\omega'|^2} = \frac{|\tau|^2}{|\omega|^4} |1 + a\omega\tau|,$$

so

$$c \leq \sqrt{c} \leq \left| \frac{\tau}{\omega^2} \right| \sqrt{|1 + a\omega\tau|}.$$

Then in a small enough neighborhood of 0 and for small enough c , we have

$$c \leq \left| \frac{\tau}{\omega^2} \right|.$$

Thus, B' is backward invariant in a small enough neighborhood of 0.

Having confined the critical set to a backward invariant set $B \cup B'$, we may, for some $\varepsilon > 0$, define a neighborhood of $L_0 \setminus \{0, [1 : 0 : 0]\}$,

$$\Omega_\varepsilon := \{|w| < \varepsilon\} \setminus B \cup B',$$

on which we can define a foliation.

Lemma 4.1.2 *For $x \in \Omega_\varepsilon$, the vertical cone field $K_x^v = \{(u, v) \in T_x\mathbb{C}^2 : |v| \geq \alpha(x)|u|\}$, where $\alpha(x) = |z/a|$, is backward invariant.*

Let \mathcal{F} be a proper vertical foliation of $L_0 \setminus \{0, [1 : 0 : 0]\}$ with respect to some cone field. See Figure 4.1.

Proof We will prove the horizontal cone $K_x^h = \{(u, v) \in T_x\mathbb{C}^2 : |v| \leq \alpha(x)|u|\}$ is forward invariant. For $x, f(x) \in \Omega_\varepsilon$, we have

$$D(f_\lambda)_x = \begin{bmatrix} 2z & -a \\ 0 & 2w \end{bmatrix}, \text{ so} \quad D(f_\lambda)_x \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2zu - av \\ 2wv \end{bmatrix}. \quad (4.3)$$

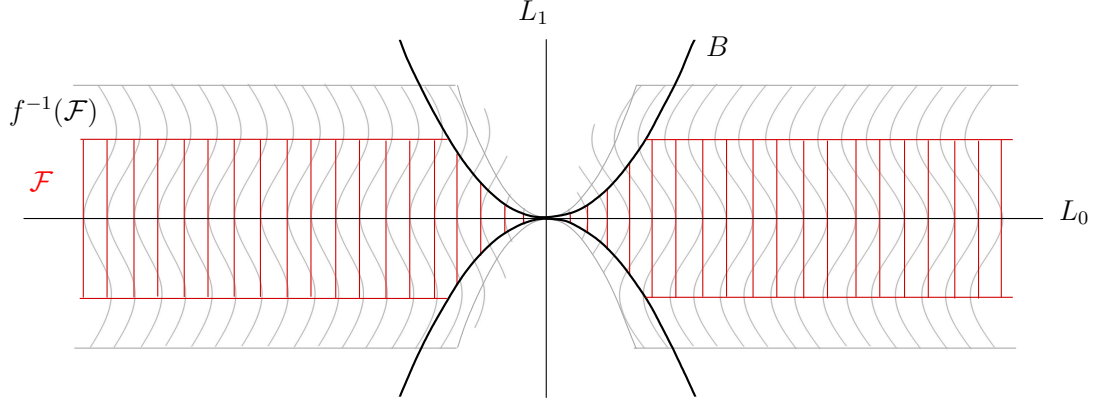


Figure 4.1. Ω_ε in a neighborhood of 0

Since $x \in \Omega_\varepsilon$, we may assume $|w| \leq c|z|^2$, where c is the constant defined in Lemma 4.1.1. Moreover, since the cone K_x is horizontal, we have $|v| \leq \alpha(x)|u|$. Thus, it suffices to show $|wv| \leq \alpha(f(x))|zu - v|$. Since $\alpha(x) = |z/a|$,

$$1 \leq \left| \frac{1}{|a|c} - 1 \right| \leq \left| \left| \frac{z^2}{aw} \right| - 1 \right| \leq \left| \frac{z^2}{aw} - 1 \right| \leq \frac{|z^2 - aw|/|a|}{|w|} = \frac{\alpha(f(x))}{|w|}. \quad (4.4)$$

Using this, we have

$$1 \leq \frac{\alpha(f_\lambda(x))}{|w|} = \frac{\alpha(f_\lambda(x))}{|w|} \left| \frac{|z|}{\alpha(x)} - 1 \right| \leq \alpha(f_\lambda(x)) \left| \frac{zu}{wv} - \frac{1}{w} \right| = \frac{\alpha(f_\lambda(x))|zu - v|}{|wv|}. \quad (4.5)$$

■

4.2 Pulling Back the Foliation

Lemma 4.2.1 *The intersected preimage $f_\lambda^{-1}(\mathcal{F}) \cap \Omega_\varepsilon$ is a proper vertical holomorphic foliation.*

Proof Let $\gamma_z \in \mathcal{F}$, so γ_z is a holomorphic disc through some point $z \in L_0 \setminus \{0, [1 : 0 : 0]\}$, that is γ_z is a submanifold of \mathbb{C}^2 of complex codimension 1. Note that for any $x \in \Omega_\varepsilon \setminus L_0$, $D(f_\lambda)_x$ has full rank since all critical curves (besides L_0) are confined

outside Ω_ε . Considering the points in L_0 , recall that $f_\lambda | L_0$ is the map $z \mapsto z^2$, so if $x \in L_0 \setminus \{0, [1 : 0 : 0]\}$, then $\text{Im}(Df_x) = L_0$. Thus, for any $x \in \gamma_z$, we have

$$\text{Im}(Df_{f^{-1}(x)}) + T_x(\gamma_z) = \mathbb{C}^2,$$

where the terms of the direct sum are linearly independent since $\text{Im}(Df_x) = L_0$ and \mathcal{F} is a vertical foliation of L_0 . Then f is transversal to L_0 , so by the preimage theorem, $f_\lambda^{-1}(\gamma_z)$ is a codimension 1 complex manifold through $f_\lambda^{-1}(z) \in L_0$.

Recall that by Lemma 4.1.1 vertical cones must be backward invariant. Thus, $f_\lambda^{-1}(\gamma_z)$, the codimension 1 complex manifold through $f_\lambda^{-1}(z) \in L_0$, is a vertical holomorphic disc provided it is bounded away from the critical set. This is achieved by intersecting $f_\lambda^{-1}(\gamma_z)$ with Ω_ε . ■

At this point, we have shown that the foliation \mathcal{F} the property that backward iterates are still vertical foliations, but we still require backward invariance of this foliation. For this, we consider the limit of backward iterates of \mathcal{F} , intersecting with Ω_ε after each iterate. That is, we define recursively

$$\mathcal{F}_n := f_\lambda^{-1}(\mathcal{F}_{n-1}) \cap \Omega_\varepsilon.$$

Lemma 4.2.2 *The sequence of backward iterates \mathcal{F}_n converge to a proper vertical holomorphic foliation, $\tilde{\mathcal{F}}$.*

Proof Let $\varphi_n: \Omega_\varepsilon \rightarrow L_0 \setminus \{0, [1 : 0 : 0]\}$ be the projection along each leaf $\gamma_z \in \mathcal{F}_n$ onto $z \in L_0$. Since vertical cone fields are backwards invariant, derivatives of φ_n are uniformly bounded. Then φ_n are locally bounded, so by Montel's theorem, $\{\varphi_n\}$ is a normal family. Thus, there is a subsequence φ_{n_k} that converges to a holomorphic map $\tilde{\varphi}$.

Let $\tilde{\mathcal{F}}$ be the foliation whose leaves are defined by $\tilde{\gamma}_z := \tilde{\varphi}^{-1}(z)$ for any $z \in L_0 \setminus \{0, [1 : 0 : 0]\}$. Since $\tilde{\varphi} | L_0 \equiv id$, any $z \in L_0 \setminus \{0, [1 : 0 : 0]\}$ is a regular value for $\tilde{\varphi}$, so by the preimage theorem, each $\tilde{\gamma}_z$ is a vertical holomorphic disc. Since $\tilde{\mathcal{F}}$ must agree with the stable foliation of S , which is backward invariant, it follows from uniqueness of holomorphic functions that $\tilde{\mathcal{F}}$ is backward invariant. ■

Then $\{\tilde{\gamma}_z \mid \tilde{\gamma}_z \cap L_0 \in S\} = W^s(S)$ must be real analytic since it is a three real dimensional topological manifold that is a restriction to leaves of a holomorphic foliation intersecting S .

5. PROOF OF THEOREM B

We'll now show that the hypothesis in Theorem A that L is transversally superattracting with degree greater than or equal to the degree of $f|L$ cannot be eliminated without adding additional hypotheses.

The Migdal-Kadanoff Renormalization map $R : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ for the Ising model on the DHL is given in homogeneous coordinates by

$$R[U : V : W] = [(U^2 + V^2)^2 : V^2(U + W)^2 : (W^2 + V^2)^2].$$

For this map, the projective line $L_0 = \{V = 0\}$ is transversally superattracting with degree 2 with R holomorphic on a forward invariant neighborhood of L_0 . Restricted to L_0 , R is given by $u \mapsto u^4$, where $u = U/W$, so $a = 2$ and $b = 4$. The invariant circle is denoted $B := \{V = 0, |u| = 1\}$. Below, we will show that $\mathcal{W}_{\text{loc}}^s(B)$ is not real analytic in the neighborhood of any point of B , thus proving Theorem B.

The second example for which $a < b$ and $W^s(S)$ is not real analytic is the following polynomial skew product of $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ given in affine coordinates by

$$f(z, w) = (z^3 + 2wz^2, w^2). \tag{5.1}$$

One can check that this map is holomorphic on a forward invariant neighborhood in \mathbb{P}^2 of the invariant line $L = \{w = 0\}$. Moreover, L is transversally superattracting with degree 2, and $f|L$ is given by $z \mapsto z^3$. Thus, $a = 2 < 3 = b$. For this map, $\mathcal{W}_{\text{loc}}^s(S)$ is not real analytic in the neighborhood of any point of S .

In this chapter, we'll provide a detailed proof of Theorem B, showing that $\mathcal{W}_{\text{loc}}^s(B)$ is not real analytic. An adaptation of the same techniques can be used to show the analogous result for the skew product f . We leave details of this adaptation to the reader.

5.1 The Migdal-Kadanoff Renormalization

In the remainder of this section, we will adopt the notation from the recent preprints [3, 4] by Bleher, Lyubich, and Roeder. Although $R : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is more convenient for illustrating Theorem A, in the proof of Theorem B it will be more convenient to work the expression of the Migdal-Kadanoff renormalization $\mathcal{R} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ in the physical coordinates (z, t) . In these coordinates, it is given by

$$(z_{n+1}, t_{n+1}) = \left(\frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2}, \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}} \right) := \mathcal{R}(z_n, t_n). \quad (5.2)$$

We consider (z, t) as affine coordinates on \mathbb{P}^2 with $z = Z/Y, t = T/Y$ for some system of homogeneous coordinates $[Z : T : Y]$. The map \mathcal{R} has an invariant projective line $\mathcal{L}_0 = \{T = 0\}$ that is transversally superattracting, except for an indeterminate point at $\mathbf{0} := [0 : 0 : 1]$, and $\mathcal{R}|_{\mathcal{L}_0}$ is given by $z \mapsto z^4$. The invariant circle is given by $\mathcal{B} = \{|z| = 1, t = 0\}$.

The map R is semi-conjugate to \mathcal{R} by means of a rational map $\Psi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\mathcal{R}} & \mathbb{P}^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{P}^2 & \xrightarrow{R} & \mathbb{P}^2 \end{array} \quad (5.3)$$

with $[U : V : W] = \Psi([Z : T : Y]) = [Y^2 : ZT : Z^2]$. The map Ψ sends \mathcal{L}_0 to L_0 , \mathcal{B} to B , and is holomorphic in a neighborhood of \mathcal{B} . Therefore, $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \Psi^{-1}(\mathcal{W}_{\text{loc}}^s(B))$. In particular, if $\mathcal{W}_{\text{loc}}^s(B)$ were real analytic in the neighborhood of any point of B , then $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ would be real analytic in the neighborhood of the preimage of that point under Ψ . So, Theorem B will follow from:

Theorem B' *The stable manifold $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic at any point.*

The reason we originally stated Theorem B for R rather than \mathcal{R} is that R is holomorphic in a full neighborhood of L_0 , so that it illustrates why the hypothesis on a and b can't be eliminated in Theorem A. One can also resolve the indeterminacy $\mathbf{0} \in \mathcal{L}_0$ for \mathcal{R} , placing it in the context of Theorem A, via a suitable birational modification (two blow-ups and one blow-down), but that is somewhat more complicated.

We will begin by proving the following proposition, and Theorem B' will follow shortly thereafter.

Proposition 5.1.1 $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic in any full neighborhood of \mathcal{B} .

This proposition will be proven by contradiction, so for the remainder of this section, we assume $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a full neighborhood of \mathcal{B} . We will begin by describing the dynamics of \mathcal{R} near \mathcal{L}_0 , and after that, with the construction of a co-dimension 1 Böttcher function φ . This is followed by the extension of the domain of φ and an exploration of the behavior of φ and \mathcal{R} in the extension. The section concludes with a proof of Proposition 5.1.1.

5.2 Dynamics in a Neighborhood of the Invariant Line

We will now briefly summarize basic properties of the dynamics for \mathcal{R} in a neighborhood of \mathcal{L}_0 from [3, Section 4].

Let $\mathbb{D}_0 := \{|z| < 1, t = 0\} \subset \mathcal{L}_0$. The orbit of any $z \in \mathbb{D}_0$ will converge to an indeterminate point $\mathbf{0} := \{(0, 0)\}$. (Informally, we will denote these points by $\mathcal{W}^s(\mathbf{0})$.) Meanwhile, points near $\mathbf{0}$ but not on \mathcal{L}_0 will converge to a superattracting fixed point $\eta := \{(0, 1)\}$.

To see what happens for large $|z|$, we write \mathcal{R} in homogeneous coordinates, obtaining

$$\mathcal{R}: [Z : T : Y] \mapsto [Z^2(Z^2 + T^2)^2 : T^2(Z^2 + Y^2)^2 : (Z^2 + T^2)(T^2Z^2 + Y^4)]. \quad (5.4)$$

There is another superattracting fixed point $\eta' := [1 : 0 : 0]$, which attracts all points of \mathcal{L}_0 with $|z| > 1$.

Lemma 5.2.1 $\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}_{\text{loc}}^s(\eta) \cup \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cup \mathcal{W}_{\text{loc}}^s(\eta')$ fills some neighborhood of $\mathcal{L}_0 \setminus \{\mathbf{0}\}$.

See [3, Lemma 4.2].

There is another invariant line $\mathcal{L}_1 := \{t = 1\}$ passing through η and η' . We have $\mathcal{R}|_{\mathcal{L}_1} : z \rightarrow z^2$.

For the remainder of this section, it will be convenient to use a system of affine coordinates centered at η' . We will use $(\lambda = Y/Z - T/Z, \tau = T/Z)$, so that $\mathcal{L}_0 = \{\tau = 0\}$ and $\mathcal{L}_1 = \{\lambda = 0\}$. In these coordinates,

$$(\lambda_{n+1}, \tau_{n+1}) = \left(\lambda_n^2 \left(\frac{\lambda_n + 2\tau_n}{1 + \tau_n^2} \right)^2, \tau_n^2 \left(\frac{1 + (\tau_n + \lambda_n)^2}{1 + \tau_n^2} \right)^2 \right) := \mathcal{R}(\lambda_n, \tau_n). \quad (5.5)$$

As before, $\mathcal{R}|_{\mathcal{L}_0} : \lambda \rightarrow \lambda^4$ and $\mathcal{R}|_{\mathcal{L}_1} : \tau \rightarrow \tau^2$.

We continue by exploring the some preliminary consequences of the hypothesis that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in such a full neighborhood of \mathcal{B} .

Proposition 5.2.1 *If $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a full neighborhood of \mathcal{B} , then there is another neighborhood Ω_0 of \mathcal{B} and a holomorphic function $\varphi : \Omega_0 \rightarrow \mathbb{C}$ such that*

- (i) *if $(\lambda, \tau) \in \Omega_0$ and $\mathcal{R}(\lambda, \tau) \in \Omega_0$, then $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$,*
- (ii) *$\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$, and*
- (iii) *$\varphi(\lambda, 0) = \lambda$.*

The function φ is analogous to the one constructed in the Proof of Theorem A'. However, Proposition 5.2.1 only gives that φ is defined on a small neighborhood of \mathcal{B} , which may not be forward invariant under \mathcal{R} .

We will exploit the fact that each $x \in B$ is hyperbolic, emitting a stable manifold $\mathcal{W}_{\text{loc}}^s(x)$ that is a one-dimensional holomorphic curve transverse to \mathcal{L}_0 . Together, the union of stable manifolds of each $x \in B$ forms a foliation of $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$; see [3, Proposition 9.2].

The notion of Levi-flat real-codimension 1 hypersurfaces $\Sigma \subset \mathbb{C}^n$ will be useful; for background see [26, 27]. A C^2 hypersurface Σ is Levi flat if through each point of Σ there is a complex codimension 1, holomorphic hypersurface. The union of these hypersurfaces is called the *Levi foliation of Σ* . Thus, the preceding paragraph shows that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is Levi flat. Note that there is another, more common but equivalent,

definition of Levi-flat given in terms of vanishing an appropriate Levi $(1, 1)$ -form [26, page 126].

Rea's Theorem [28] holds in any codimension, but here we need only

Theorem 5.2.2 (Rea) *Suppose Σ is a Levi-flat, real analytic hypersurface defined on some open $\Omega_0 \subset \mathbb{C}^n$. Then there is a neighborhood $\Omega \subset \Omega_0$ of Σ to which the Levi foliation extends uniquely and holomorphically.*

We include a sketch of the proof, as it is rather simple in this case.

In a neighborhood of any $x \in \Sigma$, we can choose a holomorphic coordinate system (u_1, \dots, u_n) such that $\Sigma = \{\text{Im}u_n = 0\}$. See [27, Remark 4.3]. In these coordinates, the Levi foliation has leaves given by $u_n = a \in \mathbb{R}$. Thus, a holomorphic extension of the foliation is obtained by letting a be complex (with small imaginary part).

To see that the extension is unique, suppose (v_1, \dots, v_n) is another holomorphic coordinate system defined in a neighborhood of x also with $\Sigma = \{\text{Im}v_n = 0\}$. In these coordinates, the Levi foliation and extension are given analogously. To see that the resulting extension is the same as that obtained using the u -coordinates, it sufficed to show that the change of coordinates $(v_1, \dots, v_n) = \psi(u_1, \dots, u_n)$ maps vertical hyperplanes $u_n = a$ to vertical hyperplanes $v_n = b$. Since $\psi(\{\text{Im}u_n = 0\}) = \{\text{Im}v_n = 0\}$, and biholomorphisms send holomorphic hypersurfaces to holomorphic hypersurfaces, this is true for any real a . Hence, it is true for any complex a (where ψ is defined).

Proof [Proof of Proposition 5.2.1] As stated above, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is foliated by a family \mathcal{F} of holomorphic stable curves at each point in \mathcal{B} , so it's Levi flat. Since $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is assumed to be real analytic, Rea's Theorem implies that this Levi foliation extends to be a complex analytic foliation in a neighborhood of $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$. Since the foliation \mathcal{F} is transverse to \mathcal{L}_0 at points of \mathcal{B} , in a small enough neighborhood $\tilde{\Omega}$, each curve γ_x of the foliation is transverse to \mathcal{L}_0 . Then we may assume Ω is the union of connected components in $\tilde{\Omega}$ of any leaf that intersects $\tilde{\Omega} \cap \{\lambda = 0\}$. Let $\varphi: \Omega \rightarrow \mathbb{C}$ be the map assigning to each $(\lambda, \tau) \in \Omega$ the point where $\gamma_{(\lambda, \tau)}$ intersects $\tau = 0$. From this, (ii)

and (iii) follow immediately. Note that it follows from a change of coordinates and the Implicit Function Theorem that φ is holomorphic.

Define Ω_0 to be the connected component of $\mathcal{R}^{-1}(\Omega) \cap \Omega$ containing \mathcal{B} . For each τ_0 with $|\tau_0|$ sufficiently small, let $\mathcal{L}_{\tau_0} := \{\tau = \tau_0\}$. Observe that $\mathcal{B}_{\tau_0} := \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cap \mathcal{L}_{\tau_0}$ is a topological circle. Since $\mathcal{B}_{\tau_0} \subset \mathcal{W}_{\text{loc}}^s(\mathcal{B})$, (i) holds on \mathcal{B}_{τ_0} and, by uniqueness properties of holomorphic functions, it holds in some open neighborhood of \mathcal{B}_{τ_0} within \mathcal{L}_{τ_0} . Varying τ_0 , these neighborhoods form an open neighborhood of \mathcal{B} contained in Ω_0 on which (i) holds. This property then extends to all of Ω_0 , since Ω_0 is connected.

■

We can suppose that the domain Ω_0 on which φ is defined, given by Proposition 5.2.1, is sufficiently small, so that it is contained in $\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}_{\text{loc}}^s(\eta) \cup \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cup \mathcal{W}_{\text{loc}}^s(\eta')$. Since \mathcal{B} has a local product structure, it is isolated in the recurrent set. Proof of this is similar to [29, Proposition 4.4]. Thus, we can choose Ω_0 smaller if necessary so that each orbit enters and leaves Ω_0 at most once.

Proposition 5.2.2 *The domain Ω_0 may be extended to Ω , a neighborhood of $\mathcal{L}_0 \setminus \{\eta', \eta\}$, such that $\varphi: \Omega \rightarrow \mathbb{C}$ is holomorphic,*

$$(i) \text{ If } (\lambda, \tau) \in \Omega \text{ and } \mathcal{R}(\lambda, \tau) \in \Omega, \text{ then } \varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4,$$

$$(ii) \mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}, \text{ and}$$

$$(iii) \varphi(\lambda, 0) = \lambda \text{ for } x \in \mathcal{L}_0 \setminus \{\eta', \eta\}.$$

In general, the push-forward of a function by a mapping is not well-defined. However, if the mapping is proper, then it is well-defined by averaging over the fibers. It was shown in [3, Sec. 4.5] that \mathcal{R} has topological degree eight. In the proposition below, we mimic this push forward under a proper mapping.

Proof Let $\Omega_n := \{x: \mathcal{R}^{-n}\{x\} \subseteq \Omega_0\}$ and C_n be the critical value set for \mathcal{R}^n . For $x \in \Omega_n \setminus C_n$, we may define

$$\varphi(x) = \frac{1}{8^n} \sum_{i=1}^{8^n} \varphi(y_i)^4, \tag{5.6}$$

where $\{y_i\}_{i=1}^{8^n} = \mathcal{R}^{-n}\{x\}$. Then locally about each $x \in \Omega_n \setminus C_n$, φ is holomorphic since each branch of \mathcal{R}^{-n} is holomorphic by the Inverse Function Theorem. If x follows a nontrivial loop around C_n , then $\varphi(x)$ has no monodromy since we are averaging over all of the fibers in (5.6). Moreover, since $|\varphi|$ is bounded on Ω_0 , (5.6) implies $|\varphi|$ is also bounded on $\Omega_n \setminus C_n$. Therefore, by the Riemann Extension Theorem, φ can be extended through the critical value curves to be holomorphic on all of Ω_n .

If $x \in \Omega_n \cap \Omega_m$ with $n \geq m \geq 0$, then $\mathcal{R}^{-n}\{x\}, \mathcal{R}^{-m}\{x\} \subset \Omega_0$. Since any orbit enters and leaves Ω_0 at most once, for any $y_i \in \mathcal{R}^{-m}\{x\}$ and each $z_j \in \mathcal{R}^{m-n}\{y_i\}$, we have that $z_j, \mathcal{R}(z_j), \dots, \mathcal{R}^{n-m}(z_j) = y_i \in \Omega$. Thus, $\varphi(y_i) = \varphi(\mathcal{R}^{n-m}(z_j)) = \varphi(z_j)^{4^{n-m}}$ since (i) holds on Ω_0 . This implies

$$\frac{1}{8^m} \sum_{y_i \in \mathcal{R}^{-m}(x)} \varphi(y_i)^{4^m} = \frac{1}{8^n} \sum_{z_j \in \mathcal{R}^{-n}(x)} \varphi(z_j)^{4^n},$$

so that the two definition of φ agree in $\Omega_n \cap \Omega_m$.

We obtain a well-defined holomorphic function φ on

$$\Omega_\infty := \bigcup_{n=0}^{\infty} \Omega_n. \quad (5.7)$$

Then we define Ω to be the connected component of $\mathcal{R}^{-1}(\Omega_\infty) \cap \Omega_\infty$ containing \mathcal{B} . Now (i) holds on all of Ω using the exactly the same proof as in Proposition 5.2.1.i.

Since \mathcal{L}_0 is forward invariant, Ω_0 intersects \mathcal{L}_0 , and $\mathcal{R}|_{\mathcal{L}_0}$ is $\lambda \mapsto \lambda^4$, it follows that Ω contains $\mathcal{L}_0 \setminus \{\eta', \eta\}$. The fact that $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$ also follows from the fact that $\Omega_0 \subset \Omega$. ■

5.3 Local Properties Near the Fixed Point

In order to study the geometry of the extended domain Ω and the properties of φ , several technical results about the dynamics near η' will be required. We may choose $\varepsilon > 0$ sufficiently small so that the bidisk

$$X_\varepsilon := \{|\lambda| < \varepsilon, |\tau| < \varepsilon\}, \quad (5.8)$$

is forward invariant, and \mathcal{R} strictly decreases each component in modulus. We continue by describing the trajectory of orbits as they converge to η' .

Proposition 5.3.1 *If $\varepsilon > 0$ is sufficiently small, then for any $\gamma \in \mathbb{Z}_+$, if $(\lambda_0, \tau_0) \in X_\varepsilon \setminus \mathcal{L}_0$, then $|\lambda_n|/|\tau_n|^\gamma \rightarrow 0$.*

This proposition implies that any point near η' and not on \mathcal{L}_0 converges to η' with an arbitrarily high degree of tangency to \mathcal{L}_1 .

Proof We first proof the proposition when $|\lambda_0| \leq |\tau_0|^\gamma$. Let $w_n := \lambda_n/\tau_n^\gamma$, so that

$$w_{n+1} = \frac{\lambda_{n+1}}{\tau_{n+1}^\gamma} = \frac{\lambda_n^2}{\tau_n^{2\gamma}} \left(\frac{(1 + \tau_n^2)^{\gamma-1} (\lambda_n + 2\tau_n)}{(1 + (\tau_n + \lambda_n)^2)^\gamma} \right)^2 = w_n^2 \tau_n^2 \left(\frac{(1 + \tau_n^2)^{\gamma-1} (2 + w_n \tau_n^{\gamma-1})}{(1 + \tau_n(1 + w_n \tau_n^\gamma)^2)^\gamma} \right)^2. \quad (5.9)$$

In the (τ, w) coordinates, $(0, 0)$ is a superattracting fixed point for \mathcal{R} . Then there is a $\delta > 0$ such that any point with $|\tau|, |w| < \delta$ is in $\mathcal{W}^s((0, 0))$. The closed disk $\{\tau = 0, |w| \leq 1\}$ collapses to $(0, 0)$. By continuity, there exists $\varepsilon > 0$ such that

$$\mathcal{R}(\{|\tau| < \varepsilon, |w| \leq 1 + \varepsilon\}) \subset \{|\tau|, |w| < \delta\} \subset \mathcal{W}^s((0, 0)). \quad (5.10)$$

Thus, for $(\lambda_0, \tau_0) \in X_\varepsilon$ with $\varepsilon > 0$ sufficiently small, if $|\lambda_0| \leq |\tau_0|^\gamma$, then the result follows.

Now it suffices to show that if $\tau_0 \neq 0$, then there is some $N \geq 0$ so that $|\lambda_n| \leq |\tau_n|^\gamma$ for any $n \geq N$. Let

$$M_1 = \min_{(\lambda, \tau) \in \overline{X_\varepsilon}} \left| \frac{1 + (\tau + \lambda)^2}{1 + \tau^2} \right|^2 \quad \text{and} \quad M_2 = \max_{(\lambda, \tau) \in \overline{X_\varepsilon}} 9 \left| \frac{1}{1 + \tau^2} \right|^2. \quad (5.11)$$

As long as $|\lambda_n| \geq |\tau_n|^\gamma$, we have

$$|\tau_{n+1}| \geq M_1 |\tau_n|^2 \quad \text{and} \quad |\lambda_{n+1}| \leq M_2 |\lambda_n|^{2+2/\gamma}. \quad (5.12)$$

This implies that

$$|\tau_n| \geq A_1 \rho_1^{2^n} \quad \text{and} \quad |\lambda_n| \leq A_2 \rho_2^{(2+2/\gamma)^n} \quad (5.13)$$

for some $A_i > 0$ and $0 < \rho_i < 1$. Then

$$\frac{|\lambda_n|}{|\tau_n|^\gamma} \leq \frac{A_2 \rho_2^{(2+2/\gamma)^n}}{A_1 \rho_1^{\gamma 2^n}} = A \rho_2^{(2+2/\gamma)^n - a\gamma 2^n} \rightarrow 0, \quad (5.14)$$

where $\rho_1 = \rho_2^a$ and $A = A_2/A_1$. Thus, for some iterate m , we have $|\lambda_m| \leq |\tau_m|^\gamma$. ■

Consider the “bullet-shaped” regions $B_{\gamma,c} := \{(\lambda, \tau) : |\lambda| \geq c|\tau|^\gamma\}$, and let $B_\gamma \equiv B_{\gamma,1}$. We will use the following horizontal and vertical cones:

$$C^h := \{|\tau| \leq |\lambda|\} \quad \text{and} \quad C^v := \{|\tau| \geq |\lambda|\}, \quad (5.15)$$

noting that $C^h = B_1$.

Corollary 5.3.1 *If $\varepsilon > 0$ is sufficiently small, then for any $\gamma \in \mathbb{Z}_+$,*

$$\mathcal{R}^{-1}(B_\gamma) \cap X_\varepsilon \subset B_\gamma.$$

Corollary 5.3.2 *For any $\gamma \in \mathbb{Z}_+$, $\bigcap_{n=0}^{\infty} \mathcal{R}^{-n}(B_\gamma) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon$*

Lemma 5.3.3 *For any sufficiently small $\varepsilon > \sigma > 0$ and any $\gamma \in \mathbb{Z}_+$, there exist $m \in \mathbb{Z}_+$ such that $\mathcal{R}^{-m}(B_\gamma) \cap (\overline{X}_\varepsilon \setminus X_\sigma) \subset C^h$.*

Proof Consider the compact set $K := (\overline{X}_\varepsilon \setminus X_\sigma) \cap C^v$. It suffices to prove that there exists $m \in \mathbb{Z}_+$ such that $\mathcal{R}^m(K) \subset X_\varepsilon \setminus B_\gamma$. By the proof of Proposition 1.6, for each $x \in K$, there exists m_x such that for any $m \geq m_x$, $\mathcal{R}^m x \in X_\varepsilon \setminus B_\gamma$, which is open. Then there is an open neighborhood U_x of x such that $\mathcal{R}^{m_x}(U_x) \subset X_\varepsilon \setminus B_\gamma$. Since K is compact, there exists m such that for any $x \in K$, $\mathcal{R}^m(x) \in X_\varepsilon \setminus B_\gamma$. ■

Recall that $\mathcal{R}|_{\mathcal{L}_0}$ is $\lambda \mapsto \lambda^4$ and $\mathcal{R}|_{\mathcal{L}_1}$ is $\tau \mapsto \tau^2$. The following distortion estimates allow local approximation of these properties near η' . Also, recall the notation $(\lambda_n, \tau_n) = \mathcal{R}^n(\lambda_0, \tau_0)$. Lastly, given two sequences x_n and y_n , we will use $x_n \asymp y_n$ to mean that $a \leq |x_n/y_n| \leq A$ for some constants $0 < a < A$.

Proposition 5.3.2 *For $\varepsilon > 0$ sufficiently small and any $\gamma \geq 1$,*

- (i) *If $(\lambda_i, \tau_i) \in B_\gamma \cap X_\varepsilon$ for $i = 0, \dots, n$, then $|\lambda_n| \asymp |\lambda_0|^{4^n}$.*
- (ii) *If $(\lambda_i, \tau_i) \in X_\varepsilon \setminus B_\gamma$ for $i = 0, \dots, n$, then $|\tau_n| \asymp |\tau_0|^{2^n}$.*

Proof Let

$$A_i = \frac{1}{|\lambda_{n-i}|^2} \left| \frac{\lambda_{n-i} + 2\tau_{n-i}}{1 + \tau_{n-i}^2} \right|^2 \leq 1 + 5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right|, \quad (5.16)$$

so that $|\lambda_{n-i+1}| = A_i |\lambda_{n-i}|^4$. Inductively, we have

$$|\lambda_n| = \left(\prod_{i=1}^n A_i^{4^{i-1}} \right) |\lambda_0|^{4^n}. \quad (5.17)$$

Recall the constants $M_1 \leq 1 \leq M_2$ from the proof of Proposition 5.3.1, which are independent of γ . We have $|\tau_n| \geq (M_1 |\tau_{n-1}|)^{2^i}$ and $|\lambda_n| \leq (M_2 |\tau_{n-1}|)^{(2+2/\gamma)^i}$, so it follows that

$$\left| \frac{\tau_n}{\lambda_n} \right| \geq \frac{M_1^{2^i}}{M_2^{(2+2/\gamma)^i}} \left| \frac{\tau_{n-1}}{\lambda_{n-1}^{(1+1/\gamma)^i}} \right|^{2^i}. \quad (5.18)$$

This implies there is a $0 < \delta < 1$ such that

$$5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right| \leq 5 (M_2 |\lambda_{n-i}|)^{(1+1/\gamma)^{i-1}} \frac{M_2}{M_1} \left| \frac{\tau_n}{\lambda_n} \right|^{1/2^i} \leq \delta^{(1+1/\gamma)^i}, \quad (5.19)$$

since M_2 is a fixed constant, $|\lambda_{n-i}| < \varepsilon$, and we can choose ε as small as we like.

It suffices to find uniform constants to estimate the product $\prod_{i=1}^n A_i^{4^{i-1}}$ independent n . Observe

$$\prod_{i=1}^n A_i^{4^{i-1}} \leq \prod_{i=1}^n \left(1 + 5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right| \right)^{4^{i-1}} \leq \prod_{i=1}^{\infty} \left(1 + \delta^{(1+1/\gamma)^i} \right)^{4^{i-1}}, \quad (5.20)$$

where the last product converges since

$$\sum_{i=1}^{\infty} 4^{i-1} \log \left(1 + \delta^{(1+1/\gamma)^i} \right) \quad (5.21)$$

converges. Thus, there is a constant A such that for any n , $\prod_{i=1}^n A_i^{4^{i-1}} \leq A$.

A similar calculation can be done to find a uniform lower bound for the product. Moreover, the proof for the vertical distortion control is similar (and easier). ■

Consider $\mathcal{R}^*(\mathcal{L}_1)$, the pullback of the curve $\mathcal{L}_1 = \{T = Y\}$, given by

$$-Z^2(T - Y)^2(T + Y)^2 = 0. \quad (5.22)$$

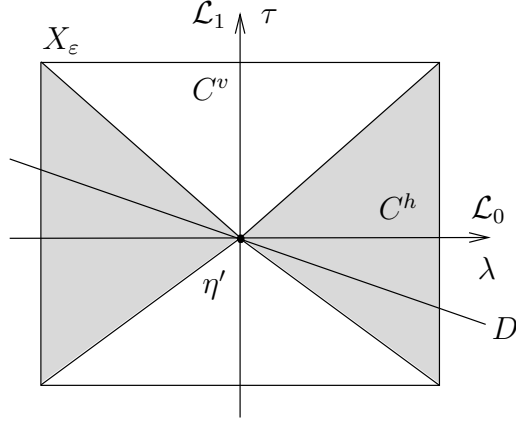


Figure 5.1. Bidisk neighborhood of η'

The pullback of \mathcal{L}_1 contains \mathcal{L}_1 , $\{Z = 0\}$, and $\{T + Y = 0\}$ (each counted with multiplicity two). Call this last curve D , so in (λ, τ) coordinates,

$$D := \{\lambda + 2\tau = 0\}. \quad (5.23)$$

Lemma 5.3.4 *If $x \in X_\varepsilon \setminus B_3$ and ε is sufficiently small, then $\mathcal{R}^{-1}\{x\} \cap C^h \neq \emptyset$ and $\mathcal{R}^{-1}\{x\} \cap C^v \neq \emptyset$.*

Proof Let $N := \{|\lambda| < \frac{1}{2}|\tau|^2\}$, and note that if $x \in X_\varepsilon \setminus B_3$, then $x \in N \cap X_\varepsilon$. Suppose $x \in N \cap X_\varepsilon$ and let $(\lambda, \tau) \in \mathcal{R}^{-1}\{x\}$. Recall that the line $D := \{\lambda + 2\tau = 0\}$ has $\mathcal{R}(D) = \mathcal{L}_1$. Also, note that N is the union over $|c| \leq 1/2$ of the curves $P_c := \{\lambda = c\tau^2\}$, and the preimage of any of these curves, $\mathcal{R}^{-1}(P_c)$, is the set of points satisfying

$$\lambda^2 \left(\frac{\lambda + 2\tau}{1 + \tau^2} \right)^2 = c\tau^4 \left(\frac{1 + (\lambda + \tau)^2}{1 + \tau^2} \right)^4. \quad (5.24)$$

It follows that if $\varepsilon > 0$ is small enough that $\left| \sqrt{c} \frac{(1 + (\lambda + \tau)^2)^2}{1 + \tau^2} \right| \leq 1$, then $\mathcal{R}^{-1}(P_c)$ is a set of points that satisfies

$$\left| \frac{\lambda}{\tau} \right| \frac{|\lambda + 2\tau|}{|\tau|} \leq 1. \quad (5.25)$$

Since the curve P_c is tangent to \mathcal{L}_1 and $\mathcal{R}(D \cup \mathcal{L}_1) = \mathcal{L}_1$, $\mathcal{R}^{-1}(P_c)$ must have a branch tangent to \mathcal{L}_1 and another branch tangent to D . Moreover, by (5.25), these preimage curves must be contained in C^v and C^h respectively. Thus, there is a preimage in C^h and another in C^v . \blacksquare

With a small amount of additional work, one can show that any point $x \in X_\varepsilon$ with ε sufficiently small has a preimage under the second iterate of \mathcal{R} contained in $C^h \cap X_\varepsilon$.

Lemma 5.3.5 *For any sufficiently small $\varepsilon > 0$ and any $k \in \mathbb{Z}_+$, there exist $\sigma > 0$ and $\gamma \in \mathbb{Z}_+$ such that if $x \in X_\sigma \setminus B_\gamma$, then x has a preorbit $\{x_{k,i}^v\}_{i=1}^k$ of length at least k contained in $C^v \cap X_\varepsilon$.*

Proof Let $\mathcal{R}(\lambda, \tau) = (\lambda', \tau') \in X_\sigma \setminus B_\gamma$, so there is a $\delta_1 > 0$ such that

$$1 \geq \frac{|\lambda'|}{|\tau'|^\gamma} \geq \frac{|\lambda|^2}{|\tau|^{2\gamma}} |\lambda + 2\tau|^2 (1 - \delta_1)^{2(\gamma-1)}. \quad (5.26)$$

For large enough γ and small enough σ , Lemma 5.3.4 implies there is some preimage $(\lambda, \tau) \in C^v$. Then $|\tau| \leq |2\tau + \lambda|$, so

$$1 \geq \frac{|\lambda|}{|\tau|^{\gamma-1}} (1 - \delta_1)^{\gamma-1}. \quad (5.27)$$

There are δ_i for $i = 2, \dots, \gamma - 2$ so that after repeating this process, we have $\mathcal{R}^{\gamma-2}(\lambda_0, \tau_0) \in X_\varepsilon \setminus B_\gamma$ with

$$1 \geq \frac{|\lambda_0|}{|\tau_0|^3} (1 - \delta_1)^{\frac{\gamma-1}{2\gamma-4}} (1 - \delta_2)^{\frac{\gamma-2}{2\gamma-3}} \cdots (1 - \delta_{\gamma-2})^{\frac{4}{2}} (1 - \delta_{\gamma-3})^3. \quad (5.28)$$

Pick σ small enough and $\gamma \geq k + 3$ so that (5.28) implies $(\lambda_0, \tau_0) \in C^v \cap X_\sigma$ and $\mathcal{R}^{-k}\{x\} \subset X_\varepsilon$. \blacksquare

Lemma 5.3.6 *For any $\gamma \in \mathbb{Z}_+$, there exists $\sigma > 0$ such that $B_\gamma \cap X_\sigma \subset \Omega$.*

Proof By Proposition 5.2.2, Ω contains some neighborhood of $\mathcal{L}_0 \setminus \{\eta', \eta\}$. By Lemma 5.3.2, there exists $\varepsilon > 0$ sufficiently small so that for any $\gamma \in \mathbb{Z}_+$, the

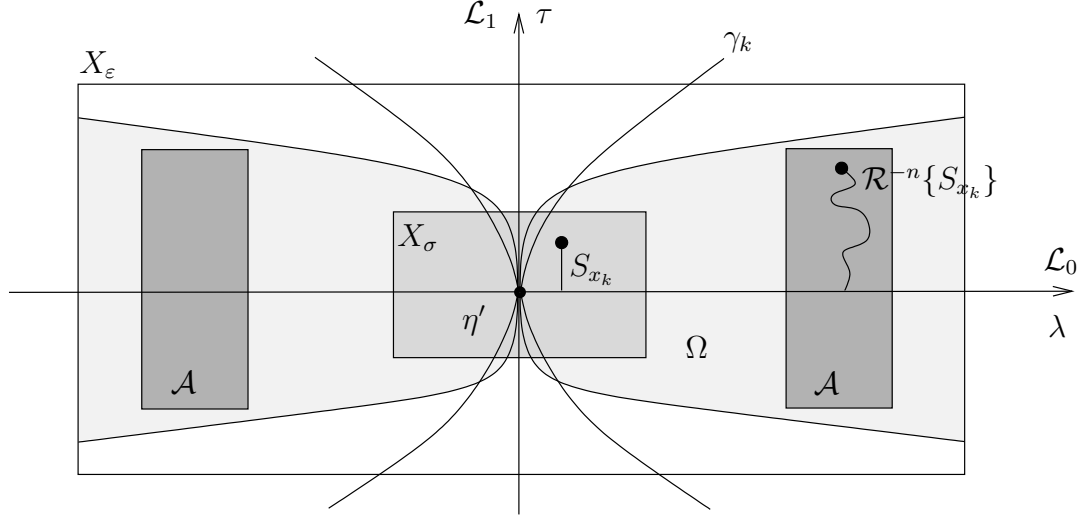


Figure 5.2. X_σ (medium gray), \mathcal{A} (dark gray), and Ω (light gray)

horizontal distortion estimates can be applied in $B_\gamma \cap X_\varepsilon$. Let $\mathcal{A} := \{a\varepsilon^{4j+2} < |\lambda| < 2A\varepsilon^{4j}, |\tau| < \delta\}$, where a and A are the constants from the distortion estimate, $j \in \mathbb{Z}_+$ is chosen so that $\mathcal{A} \subset X_\varepsilon$, and $\delta < 0$ is chosen small enough so that $\mathcal{A} \subset \Omega$. See Figure 5.2.

Let $x = (\lambda_0, \tau_0) \in B_\gamma \cap X_\sigma$ and S_x be the real straight line path connecting x to $(\lambda_0, 0) \in \mathcal{L}_0$. If $\sigma < \varepsilon$ is sufficiently small, then by Corollary 5.3.2 and the horizontal distortion estimates, there is an integer n such that both $\mathcal{R}^{-n}\{S_x\}, \mathcal{R}^{-n+1}\{S_x\} \subset \mathcal{A}$. Then $S_x \subset \Omega_\infty$ and $S_x \subset \mathcal{R}^{-1}(\Omega_\infty)$, and since S_x is connected and intersects $(\mathcal{L}_0 \setminus \{\eta', \eta\}) \subset \Omega$, we have that $x \in S_x \subset \Omega$. ■

Proposition 5.3.3 *For any sequence $\{x_m\} \subset \Omega$, if $x_m \rightarrow \eta'$, then $\varphi(x_m) \rightarrow 0$.*

Proof By Lemma 5.3.6, there exists $\sigma > 0$ such that $B_3 \cap X_\sigma \subset \Omega$. By the uniformity of φ on compact sets and the fact that $\varphi|_{\mathcal{L}_0} = id$, if $\delta > 0$ small enough, then $\mathcal{A} := \{\sigma^{4^2} < |\lambda| < \sigma, |\tau| < \delta\} \subset B_3$, and $|\varphi(x)| < 2\sigma$ for $x \in \mathcal{A}$. By Lemma 5.3.4, there is a point in the preimage of each $x_m \in X_\sigma \setminus B_3$ contained in B_3 , and Corollary 5.3.1, B_3 is backward invariant. Thus, there is a backward orbit of each x_m that

remains in $B_3 \subset \Omega$. Let $\{x_{m,n}\}$ be this preorbit. If x_m sufficiently close to η' , then by Corollary 5.3.2 there is an $N(m)$ such that $x_{m,N(m)} \in \mathcal{A}$. Using the invariance $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$, we have

$$|\varphi(x_m)| = |\varphi(x_{m,N})^{4^N}| < (2\sigma)^{4^N}. \quad (5.29)$$

As m goes to infinity, we need N to go to infinity as well in order for $x_{m,N}$ to remain in \mathcal{A} . This implies that the $\lim_{m \rightarrow \infty} |\varphi(x_m)| = 0$. ■

5.4 Proof of Non-analyticity

Proposition 5.4.1 *For any $\varepsilon > 0$ sufficiently small, there is a sequence $\{x_k\}$ converging to η' such that for each k , x_k has a preorbit of length k contained in $C^v \cap X_\varepsilon$ and a preorbit of length k contained in $C^h \cap X_\varepsilon$. Moreover, any preimage of x_k that is in X_ε is in Ω .*

Proof By Lemma 5.3.6, there exists $\varepsilon > 0$ sufficiently small so that $X_\varepsilon \cap C^h \subset \Omega$. For each $k \in \mathbb{Z}_+$, we do the following. Using Lemma 5.3.5, there exists $\gamma \in \mathbb{Z}_+$ and $\sigma > 0$ such that $x_k \in X_\sigma \setminus B_\gamma$ has a preorbit $x_{k,i}^v \subset C^v$ of length at least k . Supposing that σ is smaller if necessary, we can assure that $\mathcal{R}^{-k}\{x_k\} \subset X_\varepsilon$. Requiring that $\gamma \geq 3$, Lemma 5.3.4 implies that x_k has a first preimage, $x_{k,1}^h$, in C^h . Since C^h is backward invariant by Corollary 5.3.1, x_k has a preorbit $x_{k,i}^h \subset C^h$ of length at least k .

It remains to show that any preimage of x_k that is in X_ε is in Ω . First note that by Lemma 5.3.6, we can choose σ smaller if necessary so that $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$. By Lemma 5.3.3, there is an $m \in \mathbb{Z}_+$ such that $\mathcal{R}^{-m}(B_{\gamma+1}) \cap (X_\varepsilon \setminus X_\sigma) \subset C^h$. Let $0 < \tilde{\sigma} < \sigma$ be sufficiently small that if $x \in X_{\tilde{\sigma}}$, then $\mathcal{R}^{-m}\{x\} \subset X_\sigma$. Let $x_k \in (B_{\gamma+1} \setminus B_\gamma) \cap X_{\tilde{\sigma}}$. Using that $B_{\gamma+1}$ is backward invariant, any preimage of x_k that is in X_σ will be in $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$. Meanwhile, by the choice of $\tilde{\sigma}$, any preimage that is in $X_\varepsilon \setminus X_\sigma$ will be in $X_\varepsilon \cap C^h \subset \Omega$. ■

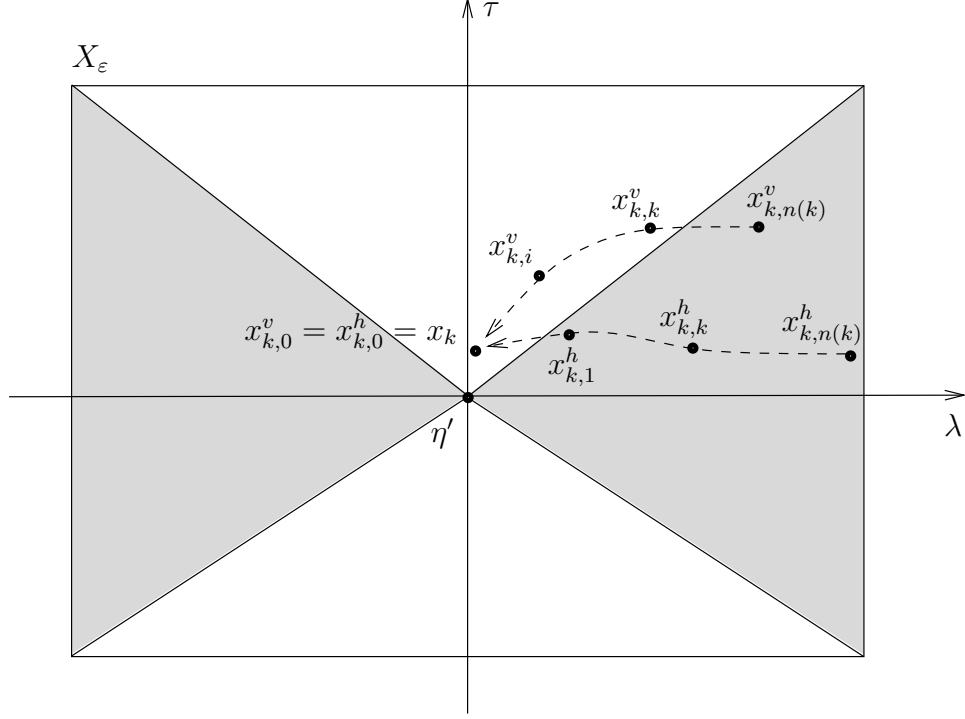


Figure 5.3. The preorbits $\{x_{k,i}^v\}$ and $\{x_{k,i}^h\}$

Proof [Proof of Proposition 5.1.1] Let $\{x_k\} \subset \Omega$ be a sequence as described in Proposition 5.4.1, and for each k , let $\{x_{k,i}^v\}_{i=1}^k \subset C^v$ and $\{x_{k,i}^h\}_{i=1}^k \subset C^h$ be preorbits of length k such that $x_{k,0}^v = x_{k,0}^h = x_k$. Each preorbit $\{x_{k,i}^h\}_{i=1}^k$ can be extended to a preorbit $\{x_{k,i}^h\}_{i=1}^{n(k)}$ with the element $x_{k,n(k)}^h$ being the last preimage remaining in X_ε . See Figure 5.3. Note that by Proposition 5.4.1 for any $0 \leq i \leq n(k)$, we have both $x_{k,i}^v, x_{k,i}^h \in \Omega$.

We first show there is a subsequence of $\{x_{k,n(k)}^h\}$, that converges to a point in $\mathcal{L}_0 \setminus \{\eta', \eta\}$. By construction, $x_{k,n(k)}^h$ is a preimage of $x_{k,1}^h \in C^h$, so

$$x_{k,n(k)}^h \in \bigcap_{i=0}^{n(k)-1} \mathcal{R}^{-i}(C^h) \cap X_\varepsilon. \quad (5.30)$$

Also by construction, $x_{k,n(k)}^h \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$, which has compact closure. Thus, there is some subsequence such that $x_{k_j,n(k_j)}^h \rightarrow x_*$ with

$$x_* \in \bigcap_{i=0}^{\infty} \mathcal{R}^{-i}(B_{\gamma_k}) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon. \quad (5.31)$$

However, since each $x_{k,n(k)}^h \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$, we must have $|x_*| \geq \varepsilon^4$.

By the vertical and horizontal distortion estimates in Proposition 5.3.2, preimages of x_k are escaping X_ε faster along $x_{k,i}^h$ than $x_{k,i}^v$, so we also have $x_{k,n(k)}^v \subset X_\varepsilon$. Note that $x_{k,i}^v$ may be in C^h for $k \leq i \leq n(k)$. Then using both vertical and horizontal distortion, there is a constant A so that

$$\text{dist}(x_{k,n(k)}^v, \eta') \leq A \text{dist}(x_k, \eta')^{\frac{1}{2^k 4^{n-k}}} \asymp A \text{dist}(x_{k,n(k)}^h, \eta')^{\frac{4^n}{2^k 4^{n-k}}} \leq A\varepsilon^{2^k}, \quad (5.32)$$

which converges to 0 as $k \rightarrow \infty$. Thus, the sequence $x_{k,n(k)}^v$ converges to η' .

By Proposition 5.3.3, $\varphi(x_{k,n(k)}^v) \rightarrow 0$ as $k \rightarrow \infty$. We also have that $|\varphi(x_{k_j,n(k_j)}^h)| \rightarrow |\varphi(x_*)| \geq \varepsilon^4$ as $k \rightarrow \infty$. However, $x_{k_j,n(k_j)}^h$ and $x_{k_j,n(k_j)}^v$ are both n th preimages of x_{k_j} , and using the invariance $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$, this implies $|\varphi(x_{k_j,n(k_j)}^v)| = |\varphi(x_{k_j,n(k_j)}^h)|$ for every $n(k)$. Then $0 = |\varphi(x_*)| \geq \varepsilon^4$, a contradiction. ■

Lemma 5.4.1 *If $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $x \in \mathcal{B} \setminus \{(\pm i, 0)\}$, then $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $\mathcal{R}(x)$.*

Proof Images of real analytic hypersurfaces under holomorphic maps were considered by Baouendi and Rothschild [30]. Suppose that M is a germ of a real analytic hypersurface in \mathbb{C}^N and H is the germ of a holomorphic map from \mathbb{C}^N to \mathbb{C}^N with $H(0) = 0$. The germ H is called *finite* if every point in some neighborhood of 0 has finitely many preimages. It is shown in [30, Theorem 4] that if H is finite and $M' := H(M)$ is smooth in some neighborhood of 0, then M' is actually real analytic.

We are in the position to apply this result, since \mathcal{R} sends $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ from the neighborhood of any $x \in \mathcal{B}$ to $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ within a smaller neighborhood of $\mathcal{R}(x)$. However, we must avoid the vertical lines $z = \pm i$, which are collapsed by \mathcal{R} to the fixed point $(1, 0) \in \mathcal{B}$. Away from these lines, \mathcal{R} is finite. ■

Proof [Proof of Theorem B'] By Proposition 5.1.1, there is some point $x \in \mathcal{B}$ at which $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic. We will now use the fact that \mathcal{R} is expanding on \mathcal{B} to show that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic in the neighborhood of any point of \mathcal{B} .

Since $\mathcal{R}|_{\mathcal{B}}$ is $z \mapsto z^4$, it is expanding on \mathcal{B} , so there is some iterate n such that $\mathcal{R}^n(U \cap \mathcal{B}) = \mathcal{B}$. Because we assumed $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at every point of $U \cap \mathcal{B}$, we can use Lemma 5.4.1 iteratively to see that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at every point of \mathcal{B} , except perhaps at the iterated images of $(\pm i, 0)$. However, these consist of just the fixed point $(1, 0)$. To see that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $(1, 0)$ note that $(1, 0)$ is also the image of $(-1, 0)$ under \mathcal{R} , where $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic. Thus, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ must be real analytic at every point of \mathcal{B} , which is impossible by Proposition 5.1.1.

We now know that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic in the neighborhood of any point of \mathcal{B} . However, it could still be real analytic in the neighborhood of some other point. We now show that this is also impossible.

Each stable manifold $\mathcal{W}_{\text{loc}}^s(x_0)$ can be expressed as the graph of a convergent power series:

$$z = h(t, z_0) = \sum_{j=0}^{\infty} a_j(z_0)t^j \quad \text{where} \quad x_0 = (z_0, 0). \quad (5.33)$$

Since each $\mathcal{W}_{\text{loc}}^s(x_0)$ depends continuously on $z_0 \in \mathcal{B}$, the coefficients $a_j(z_0)$ are continuous functions of z_0 . Therefore, there is a uniform radius of convergence $\delta > 0$. For the remainder of the proof, we suppose that the neighborhood in which $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is defined is contained in $|t| < \delta/3$.

Suppose $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a neighborhood of some x_1 . Then one can express leaves of the stable foliation near x_1 as graphs of some convergent power series

$$z = k(t, z_1) = \sum_{j=0}^{\infty} b_j(z_1)(t - t_1)^j. \quad (5.34)$$

The function $(z_1, t) \mapsto (z, t)$, with z given by (5.34), gives a parametrization of $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ near x_1 with z_1 varying over the real analytic arc $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cap \{t = t_1\}$ and t varying over some complex disc centered at t_0 . Since we have assumed $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic near x_1 , the parameterization is an analytic function. In particular, $\frac{\partial^j}{\partial t^j} z$ is real analytic

for each $j \geq 0$. Restricting to $t = t_1$ we see that each of the coefficients $b_j(z_1)$ is a real analytic function of z_1 .

We now use this to show that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is also real analytic in a neighborhood of the unique point x_0 for which $x_1 \in \mathcal{W}_{\text{loc}}^s(x_0)$. Since $\mathcal{W}_{\text{loc}}^s(x_0)$ is the graph of a holomorphic function over $|t| < \delta$, $|t_1| < \delta/3$ implies that (5.34) converges on the disc $|t - t_0| < \delta/2$. In particular, each of the holomorphic discs defined by (5.34) crosses all the way through \mathcal{B} . As they depend real analytically on z_1 , this implies that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a neighborhood of $x_0 \in \mathcal{B}$, which is not possible. ■

6. PHYSICAL INTERPRETATION

In this chapter we will relate Theorems B' to the Ising Model on the DHL. We refer the reader to [3,4] for physical background. The DHL is a sequence of graphs Γ_n obtained in a self-similar way. Associated to each graph is a partition function $Z_n(z, t)$ whose zeros

$$\mathcal{S}_n^c := \{(z, t) \in \mathbb{C}^2 : Z_n(z, t) = 0\}$$

describe the singularities of the Ising model associated to Γ_n . They are called the *Lee-Yang-Fisher zeros*. The actual physics is described by the limit $n \rightarrow \infty$. It is proved in [4] that the limiting distribution of zeros exists as a closed, positive $(1, 1)$ -current \mathcal{S}^c on \mathbb{P}^2 . In fact, $\mathcal{S}^c = \frac{1}{2}\Psi^*S$, where S is the Green current for R . The support of \mathcal{S}^c describes locus where phase transitions occur in \mathbb{C}^2 .

It is shown in [4] that at low complex temperatures $\text{supp } \mathcal{S}^c$ coincides with $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$. Combining Theorem B' with the work from [4] gives the following:

Corollary 6.0.2 *At low complex temperatures ($|t|$ small), the locus of phase transitions for the Ising model on the DHL is a 3 real-dimensional manifold that is C^∞ but not real analytic.*

A preferred subset of the Lee-Yang-Fisher zeros is obtained by requiring that $t \in [0, 1]$, which correspond to “physical” temperatures. The Lee-Yang Circle Theorem [31,32] asserts that for each n and fixed $t_0 \in [0, 1]$, zeros of partition function $Z_n(z, t_0)$ corresponding to Γ_n lie on the unit circle $\mathbb{T}_{t_0} := \{|z| = 1, t = t_0\}$. Let

$$\mathcal{C} = \{|z| = 1, t \in [0, 1]\}.$$

The *Lee-Yang zeros* are defined by

$$\mathcal{S}_n := \{(z, t) \in \mathcal{C} : Z_n(z, t) = 0\}.$$

Isakov [33] proved for any $t_0 > 0$ sufficiently small the free energy for the Ising model on the \mathbb{Z}^d lattice with $d > 1$ does not have analytic continuation through any point of the circle \mathbb{T}_{t_0} . This implies that the limiting distribution of Lee-Yang zeros for the \mathbb{Z}^d lattice with $d > 1$ does not have real analytic density in the neighborhood of any point of the circle $t = t_0$. In the remainder of this chapter, we discuss how Corollary 6.0.2 can be related to Isakov's result.

One can check that \mathcal{R} maps the Lee-Yang cylinder \mathcal{C} into itself, with the Lee-Yang zeros corresponding to Γ_{n+1} obtained by pulling back the Lee-Yang zeros corresponding to Γ_n under $\mathcal{R}|_{\mathcal{C}}$. The map $\mathcal{R}: \mathcal{C} \rightarrow \mathcal{C}$ was also studied previously by Bleher and Žalys [34].

In [3], Bleher, Lyubich, and Roeder describe the limiting distribution of Lee-Yang zeros for the DHL; let us provide a very brief summary. Let $\mathcal{C}_1 := \mathcal{C} \setminus \{t = 1\}$. It was shown that $\mathcal{R}: \mathcal{C}_1 \rightarrow \mathcal{C}_1$ is partially hyperbolic, with a unique central foliation \mathcal{F}^c which is vertical (with respect to a suitable cone field) on \mathcal{C}_1 . In particular, one can define the \mathcal{F}^c holonomy map $g_t: \mathbb{T}_0 \rightarrow \mathbb{T}_t$. The limiting distribution of Lee-Yang zeros at temperature $t_0 \in [0, 1)$ is obtained as the pushforward $\mu_t = g_{t_0*} \text{Leb}$, where Leb is the normalized Lebesgue measure on \mathbb{T}_0 .

In a neighborhood of \mathcal{B} , \mathcal{F}^c coincides with the stable foliation of \mathcal{B} , which is a union of the real analytic curves $\mathcal{W}_{\text{loc}}^s(x) \cap \mathcal{C}$, taken over $x \in \mathcal{B}$. It is shown in [4, Lemma 3.2] that the stable foliation of \mathcal{B} within \mathcal{C} has the same regularity that the stable manifold $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ does as a submanifold of \mathbb{C}^2 . (In fact, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ was shown to be a C^∞ manifold in [4] by first showing that the stable foliation of \mathcal{B} within \mathcal{C} is C^∞ .)

Therefore, Theorem B' implies that the central foliation is not real analytic at low temperatures. Moreover, by [3], an open dense set of points from \mathcal{C} have orbits converging to \mathcal{B} . Since \mathcal{F}^c is invariant, this implies the following:

Theorem 6.0.3 *\mathcal{F}^c is not real analytic in the neighborhood of any point of \mathcal{C} .*

Using the holonomy description of the limiting distribution of Lee-Yang zeros, we find the following modest analog of Isakov's Theorem for the DHL:

Corollary 6.0.4 *For any $z = e^{i\phi} \in \mathcal{B}$, there is a dense set of $t_0 \in [0, 1]$ so that the limiting distribution of Lee-Yang zeros within \mathbb{T}_{t_0} does not have real analytic density at (t_0, ϕ) .*

7. OPEN PROBLEMS

7.1 Regularity of Superstable Manifolds When b Is Larger Than a

Theorem A naturally leads one to question whether there are necessary conditions for $\mathcal{W}_{\text{loc}}^s(S)$ to be real analytic. If $a < b$, the superattracting direction is not as strong as the expansion within L ; we believe there should be a way to exploit this to answer the following question:

Question 1 *Is there a “generic” class of mappings f with $a < b$ for which $\mathcal{W}_{\text{loc}}^s(S)$ is not real analytic?*

Here we are attempting to generalize of the technique in Section 5 for the Migdal-Kadanoff renormalization that proves $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic. If f happens to be a product, the stable manifold will be real analytic, so the class of functions not producing a real analytic stable manifold is at best generic in some sense. The technique used for \mathcal{R} (given by (5.2)) and f (given by (5.1)) relies on a second invariant line L_1 such that $f|_{L_1}$ is the map $w \mapsto w^a$. We suspect one may use the degree a transversal superattraction of L to generate an invariant cone field to serve the same purpose in the general case.

Question 2 *For any a and b , is $\mathcal{W}_{\text{loc}}^s(S)$ a C^∞ manifold?*

Following the method in [3, Proposition 9.12], define the sequence $B_n(x) := \frac{1}{b^n} Df^n(x)$. It is not difficult to show B_n converges uniformly on compact subsets of $\mathcal{W}_{\text{loc}}^s(S)$ at super-exponential rate to a matrix-valued function $B(x)$. The goal is to prove this function B is C^∞ in any neighborhood of S , since one can use the invariance

$$bB_n(x) = B_{n-1}(f(x))Df(x) \text{ and } bB(x) = B(f(x))Df(x), \quad (7.1)$$

to extend the result to any compact subset of $\mathcal{W}^s(S)$. $B(x)$ and all of its derivatives are converging so fast that we believe Whitney's extension theorem could be used to extend B to a C^∞ function in a neighborhood of S . In this case, since L is transversally superattracting, $\ker B(x)$ would be a C^∞ holomorphic line field, so that one could integrate it to get a C^∞ foliation by holomorphic discs of S . Within $\mathcal{W}_{\text{loc}}^s(S)$, $\ker(B)$ is the field of tangent planes to the Levi foliation of $\mathcal{W}_{\text{loc}}^s(S)$. Therefore, $\mathcal{W}_{\text{loc}}^s(S)$ is formed as a union of the holomorphic discs from the C^∞ foliation.

The only missing piece is the application of Whitney's Extension Theorem [35] which provides a partial converse to Taylor's Theorem. Let U be an open subset of \mathbb{R}^n , and X a closed subset of U . As described in [36], Whitney's theorem asserts that a function f defined in X is the restriction of F^0 , a C^m function in U ($m \in \mathbb{N}$ or $m = +\infty$) provided there exists a sequence $(F^k)_{|k| \leq m}$ of functions defined in X which satisfies for each $|k| \leq m$,

$$(R_x^m F)^k(y) := F^k(y) - \sum_{|j| \leq m-|k|} \frac{F^{k+j}(x)}{j!} \cdot (y-x)^j = o(|x-y|^{m-|k|}). \quad (7.2)$$

Roughly speaking, one must control the tails of the Taylor expansion uniformly.

7.2 Lee-Yang Density

Recall Corollary 6.0.4. Unfortunately, the fact that \mathcal{F}^s not real analytic at any point does not imply that none of the non-trivial holomomies are real analytic. Isakov [33] proved a similar result for Ising models on the \mathbb{Z}^d lattice. However, Isakov's result required a great deal of difficult and complicated analysis. We would like to prove the analogous result:

Conjecture 1 *For temperature $0 < t < t_c$, the limiting density of Lee-Yang zeros for the DHL $\rho_t(\phi)$ is C^∞ , but not real analytic.*

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