

Results

Let $f : X \dashrightarrow X$ be a dominant meromorphic self-map of a compact connected Hermitian manifold X of dimension $n > 1$. Suppose also there is $L \subset X$, an embedded copy of \mathbb{P}^1 , with f holomorphic in a neighborhood of L , L is invariant and transversally superattracting of degree a , and $f|L$ conjugate to $z \mapsto z^b$. For such maps, the Julia set of $f|L$ is an invariant circle S , which is a hyperbolic set for f . Let N be an open neighborhood of L such $f : N \rightarrow N$ is holomorphic. Given a neighborhood Ω of S , the local stable manifold

$$\mathcal{W}_{\text{loc}}^s(S) = \{x \in N : f^n x \in \Omega \text{ and } f^n x \rightarrow S \text{ as } n \rightarrow \infty\}$$

is a real $2n - 1$ dimensional manifold.

Theorem 1: If $a \geq b$, then $\mathcal{W}_{\text{loc}}^s(S)$ has real analytic regularity.

To prove the theorem, we localized to the situation to a tubular neighborhood of L which is forward invariant under f . This tubular neighborhood is biholomorphic to a neighborhood of the zero section of the normal bundle to L . Theorem 1 is a direct consequence of the following:

Theorem 1': Consider any holomorphic \mathbb{C}^{n-1} bundle over \mathbb{P}^1 with zero section L . Let N be an open neighborhood of L and $f : N \rightarrow N$ a holomorphic map. Suppose L is invariant and transversally superattracting with degree a and that $f|L$ is conjugate to $z \mapsto z^b$ with invariant circle S . If $a \geq b$, then $\mathcal{W}_{\text{loc}}^s(S)$ has real analytic regularity.

The condition $a \geq b$ cannot be improved without additional hypotheses. Consider the Migdal-Kadanoff renormalization mapping $\mathcal{R} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$\mathcal{R}(z, t) = \left(\frac{z^2(z^2 + t^2)}{1 + t^2 z^2}, \frac{t^2(z^2 + 1)^2}{(z^2 + t^2)(1 + t^2 z^2)} \right).$$

The complex line $\mathcal{L}_0 := \{t = 0\}$ is invariant under \mathcal{R} and transversally superattracting with degree 2. Within this line, the map \mathcal{R} is given by $z \mapsto z^4$. Therefore, the circle

$$\mathcal{B} = \{|z| = 1, t = 0\} \subset \mathcal{L}_0,$$

is invariant under \mathcal{R} , attracting in the t direction, and expanding in the z direction. This situation satisfies the hypotheses for Theorem 1 with the exception that $2 = a < b = 4$. The stable manifold, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$, was previously shown to be C^∞ by Bleher, Lyubich, and Roeder. However, we proved the following:

Theorem 2: The stable manifold $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic at any point.

The skew product map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, given by

$$f(z, w) = (z^3 + 2wz^2, w^2)$$

similarly satisfies the hypotheses of Theorem 1 with the exception that $a < b$ and produces as stable manifold $\mathcal{W}_{\text{loc}}^s(S)$ that is not real analytic. This example and Theorem 2 show that the result in Theorem 1 is sharp.

Structure of $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$

Consider the natural extension

$$\hat{S} := \{(x_i)_{i \leq 0} : x_i \in S \text{ and } f(x_i) = x_{i+1}\}.$$

We'll denote such histories by $\hat{x} = (x_i)_{i \leq 0} \in \hat{S}$. Notice that the action of f naturally lifts to an action $\hat{f} : \hat{S} \rightarrow \hat{S}$.

Note that S is hyperbolic for the map f . By the stable manifold theorem each point $x \in S$ will have local stable manifold $\mathcal{W}_{\text{loc}}^s(x)$ that is a complex $n - 1$ ball holomorphically embedded into N and each prehistory \hat{x} will have a local unstable manifold $\mathcal{W}_{\text{loc}}^u(\hat{x})$, which is a holomorphic disc. They depend continuously on x and \hat{x} . (In this case, the unstable manifolds all lie in L .)

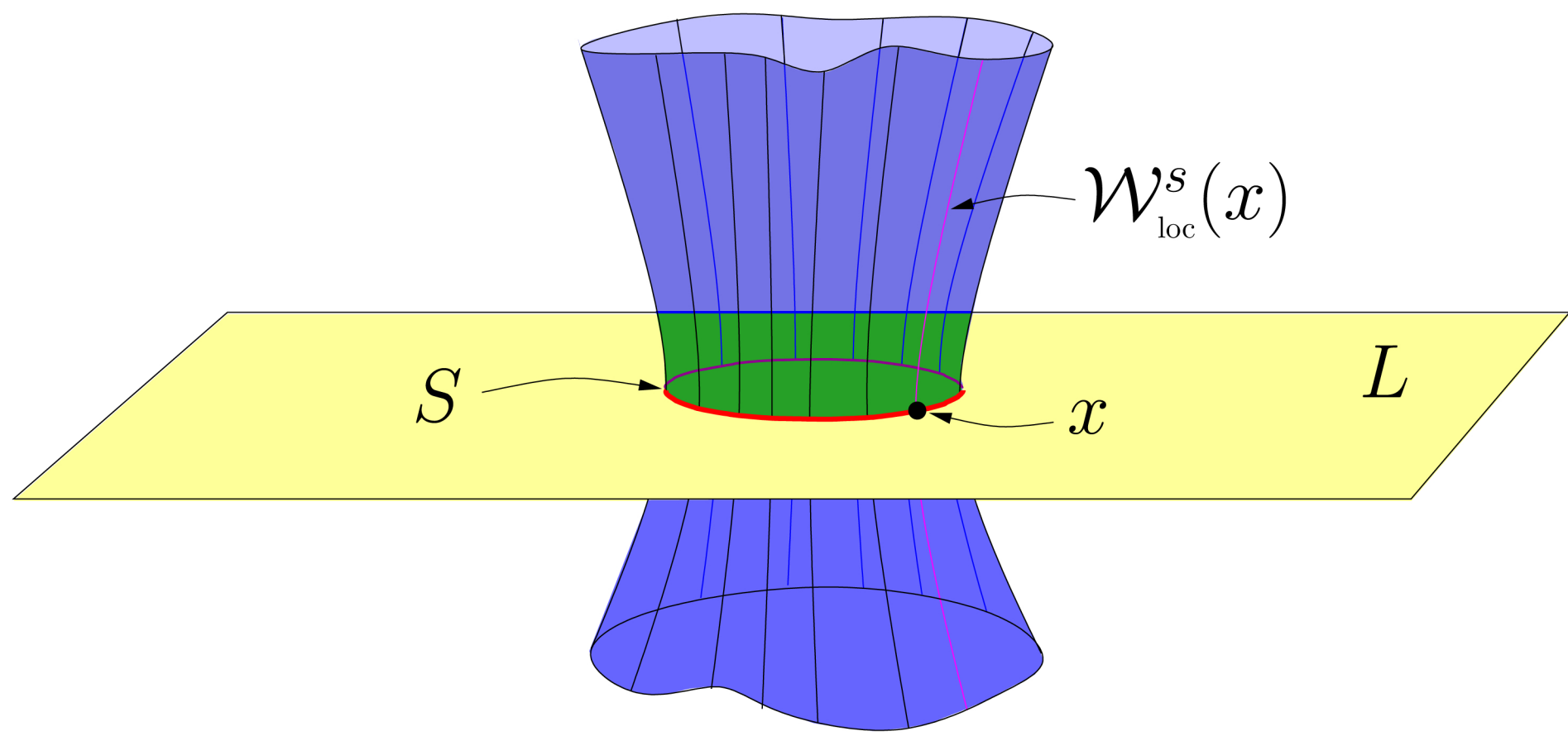
Given a neighborhood Ω of S ,

$$\mathcal{W}_{\text{loc}}^s(S) := \{x \in N : f^n x \in \Omega \text{ and } f^n x \rightarrow S \text{ as } n \rightarrow \infty\} = \bigcup_{x \in S} \mathcal{W}_{\text{loc}}^s(x)$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}_{\text{loc}}^s(S)$ means that it holds for any sufficiently small neighborhood of S).

Since S has a local product structure $\mathcal{W}_{\text{loc}}^s(S)$ is the union of the local stable manifolds $\mathcal{W}_{\text{loc}}^s(x)$ of points $x \in S$. The local stable manifolds of points are pairwise disjoint and depend continuously on the base point, therefore we have:

Corollary 3: $\mathcal{W}_{\text{loc}}^s(S)$ is a topological manifold of real dimension $2n - 1$.



General Form for f

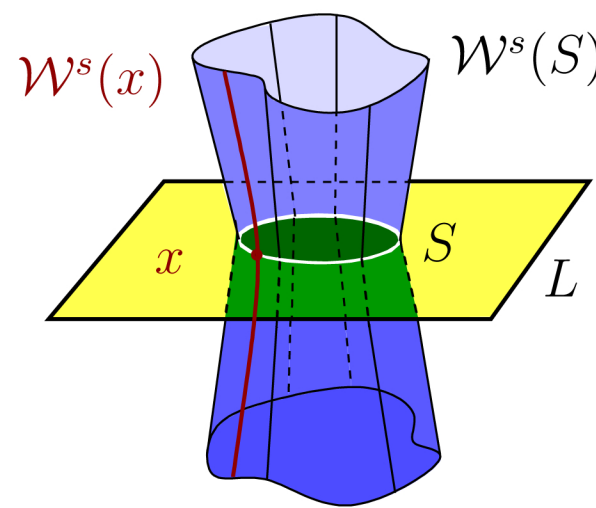
The \mathbb{C}^{n-1} bundle over \mathbb{P}^1 can be described by two systems of locally trivializing coordinates $(z, \mathbf{w}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and $(\zeta, \boldsymbol{\omega}) \in \mathbb{C} \times \mathbb{C}^{n-1}$. For $z \neq 0$, the are related by $\zeta = 1/z$ and $\boldsymbol{\omega} = A_z \mathbf{w}$, with $A_z : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ a linear isomorphism depending holomorphically on z . These trivializations can be chosen so that the dynamics on the zero section is $z \mapsto z^b$. Calculations are similar in both coordinates, so the calculations in the sketch of the proof will be shown in (z, \mathbf{w}) coordinates.

Lemma 4: There are holomorphic functions \mathbf{g}_1 and \mathbf{g}_c for each $|c| = a$ such that in the (z, \mathbf{w}) coordinates

$$f(z, \mathbf{w}) = \left(z^b + \mathbf{w} \cdot \mathbf{g}_1(z, \mathbf{w}), \sum_{|c|=a} \mathbf{w}^c \mathbf{g}_c(z, \mathbf{w}) \right).$$

Similarly, there are holomorphic functions \mathbf{h}_1 and \mathbf{h}_c for each $|c| = a$ such that in the $(\zeta, \boldsymbol{\omega})$ coordinates

$$f(\zeta, \boldsymbol{\omega}) = \left(\zeta^b + \boldsymbol{\omega} \cdot \mathbf{h}_1(\zeta, \boldsymbol{\omega}), \sum_{|c|=a} \boldsymbol{\omega}^c \mathbf{h}_c(\zeta, \boldsymbol{\omega}) \right).$$



Superstable Manifolds of Invariant Circles

Scott Kaschner and Roland Roeder

Theorem 1: Sketch of Proof

Codim. 1 Böttcher Fns.

Let $(z_n, \mathbf{w}_n) := f^n(z, \mathbf{w})$. Motivated by Böttcher's theorem, we consider a sequence of functions

$$\varphi_n(z, \mathbf{w}) = z_n^{1/b^n}.$$

We will show that the φ_n converge uniformly on compact subsets of some forward invariant neighborhood Ω of S to a holomorphic function φ that semi-conjugates f to $z \mapsto z^b$.

$$\varphi(f(z, \mathbf{w})) = \varphi(z, \mathbf{w})^b.$$

To make sense of the b^n -th roots and the limit, we'll rewrite each φ_n as telescoping product:

$$\varphi = \lim_{n \rightarrow \infty} \varphi_n = z_0 \cdot \frac{z_1^{1/b}}{z_0} \cdot \frac{z_2^{1/b^2}}{z_1^{1/b}} \cdot \frac{z_3^{1/b^3}}{z_2^{1/b^2}} \cdots = z_0 \prod_{n=0}^{\infty} \left(\frac{z_{n+1}}{z_n^b} \right)^{\frac{1}{b^{n+1}}},$$

where it follows from the general form for f that

$$\frac{z_{n+1}}{z_n^b} = \frac{z_n^b + \mathbf{w}_n \cdot \mathbf{g}_1(z_n, \mathbf{w}_n)}{z_n^b} = 1 + \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n).$$

Proposition 5: There is a forward invariant neighborhood Ω of S , if $(z, \mathbf{w}) \in \Omega \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$, then

$$\left| \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n) \right| < \frac{1}{2},$$

For points in Ω , the b^n -th root is defined by taking the branch cut along the negative real axis. Moreover, the infinite product (0.1) on Ω , since the corresponding sum of logarithms converges:

$$\sum_{n=1}^{\infty} \log \left| \frac{z_{n+1}}{z_n^b} \right|^{\frac{1}{b^{n+1}}} \leq \sum_{n=1}^{\infty} \frac{1}{b^{n+1}} \log 2.$$

Blowing Up

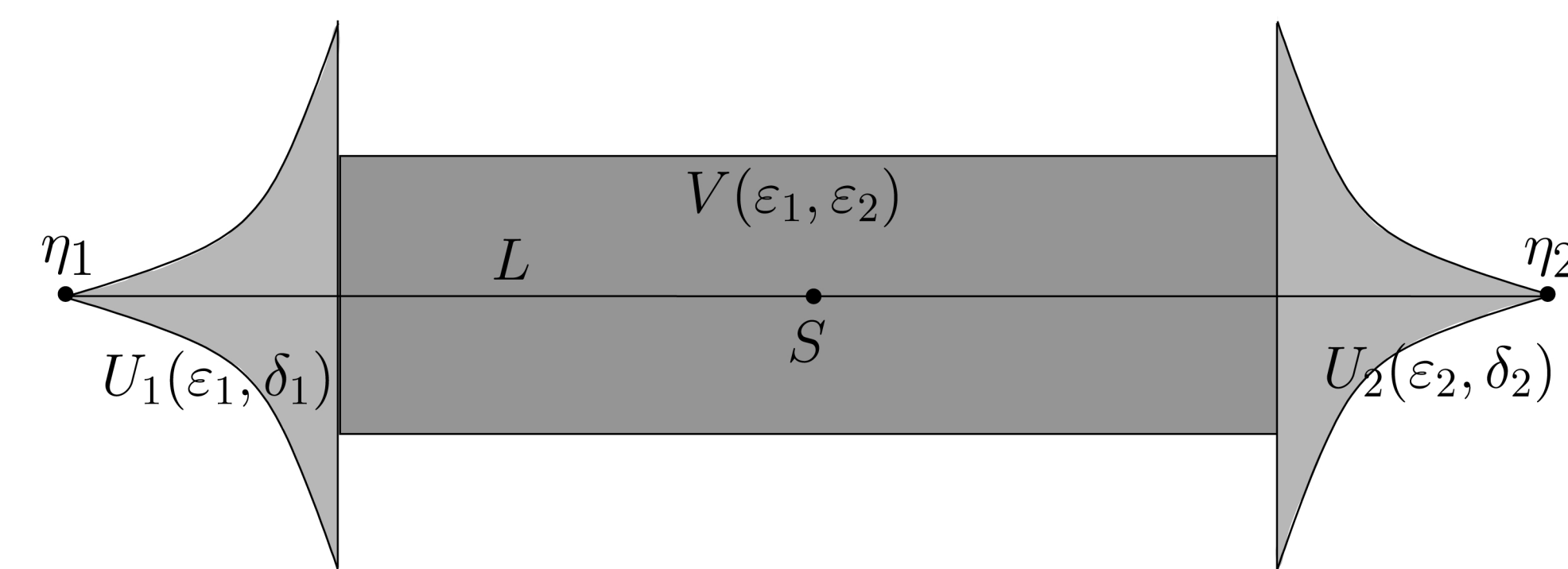
It remains only to prove Proposition 2. To construct Ω , first note that for any $K_1 > 0$ sufficiently small, $\{|\mathbf{w}| \leq K_1\} \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$ is a compact subset of \mathbb{C}^n . Since \mathbf{g}_1 is holomorphic on \mathbb{C}^n , there is a bound $|\mathbf{g}_1(z, \mathbf{w})| \leq K_2$ on any such compact set.

We will take an inductive sequence of b point blow-ups at the fixed point η_1 . Consider the system of coordinates $z, \boldsymbol{\lambda} = \frac{\mathbf{w}}{z^b}$ centered at the intersection point of $E_{\eta_1, b}$ with L_b . Let us denote $(z', \boldsymbol{\lambda}') = \tilde{f}(z, \boldsymbol{\lambda})$, where \tilde{f} is the extension of f to the final blow-up. We have

$$z' = z^b + z^b \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda})$$

$$\boldsymbol{\lambda}' = \frac{\mathbf{w}'}{(z')^d} = \frac{\sum_{|c|=a} (z^b \boldsymbol{\lambda})^c \mathbf{g}_c(z, z^b \boldsymbol{\lambda})}{(z^b + z^b \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda}))^b} = \frac{z^{b(a-b)} \sum_{|c|=a} \boldsymbol{\lambda}^c \mathbf{g}_c(z, z^b \boldsymbol{\lambda})}{(1 + \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda}))^b}.$$

Notice that this extension \tilde{f} is holomorphic in a neighborhood of $(z, \boldsymbol{\lambda}) = (0, \mathbf{0})$ and that this point is superattracting for \tilde{f} .



Thus, $\varepsilon_i, \delta_i > 0$ and V can be made small enough that $\Omega = V(\varepsilon_1, \varepsilon_2) \cup U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$ is forward invariant.

Defining Function

For any $(z, \mathbf{w}) \in \mathcal{W}_{\text{loc}}^s(S)$ we have $|\varphi(z, \mathbf{w})| = 1$ so that $\psi := \log |\varphi|$ will be a real analytic function that vanishes on $\mathcal{W}_{\text{loc}}^s(S)$. Since the derivative $D\psi$ is non-zero on S , we have that $\{\psi = 0\}$ is a real analytic $2n - 1$ real-dimensional manifold in some neighborhood of S . Recall $\mathcal{W}^s(S) \subset \{\psi = 0\}$ is also a real $2n - 1$ dimensional manifold, so by invariance of domain, $\mathcal{W}^s(S) = \{\psi = 0\}$ in this neighborhood.

Theorem 2: Sketch of Proof

Extending the Levi Foliation

\mathcal{R} can be written in new coordinates:

$$(\lambda_{n+1}, \tau_{n+1}) = \left(\lambda_n^2 \left(\frac{\lambda_n + 2\tau_n}{1 + \tau_n^2} \right)^2, \tau_n^2 \left(\frac{1 + (\tau_n + \lambda_n)^2}{1 + \tau_n^2} \right)^2 \right) := \mathcal{R}(\lambda_n, \tau_n),$$

so $\mathcal{L}_0 = \{\tau = 0\}$ and $\mathcal{L}_1 = \{\lambda = 0\}$ are invariant lines both passing through the fixed points $\eta' = (0, 0)$ and $\eta = [1 : 0 : 0]$. Also, $\mathcal{R}|_{\mathcal{L}_0} : \lambda \rightarrow \lambda^4$ and $\mathcal{R}|_{\mathcal{L}_1} : \tau \rightarrow \tau^2$.

A \mathbb{C}^2 hypersurface Σ is Levi flat if though each point of Σ there is a complex codimension 1, holomorphic hypersurface. The union of these hypersurfaces is called the *Levi foliation* of Σ . Thus, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is Levi flat.

Rea's Theorem: Suppose Σ is a Levi-flat, real analytic hypersurface defined on some open $\Omega_0 \subset \mathbb{C}^n$. Then there is a neighborhood $\Omega \subset \Omega_0$ of Σ to which the Levi foliation extends uniquely and holomorphically.

Assuming $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic for the sake of contradiction, Rea's theorem can be used to prove:

Proposition 6: If $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a full neighborhood of \mathcal{B} , then there is another neighborhood Ω_0 of \mathcal{B} and a holomorphic function $\varphi : \Omega_0 \rightarrow \mathbb{C}$ such that

- if $(\lambda, \tau) \in \Omega_0$ and $\mathcal{R}(\lambda, \tau) \in \Omega_0$, then $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$,
- $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$, and
- $\varphi(\lambda, 0) = \lambda$.

Note that φ is defined on a small neighborhood of \mathcal{B} , which may not be forward invariant under \mathcal{R} .

Proposition 7: The domain Ω_0 can be extended to Ω , a neighborhood of $\mathcal{L}_0 \setminus \{\eta', \eta\}$ on which all properties of Proposition 3 hold.

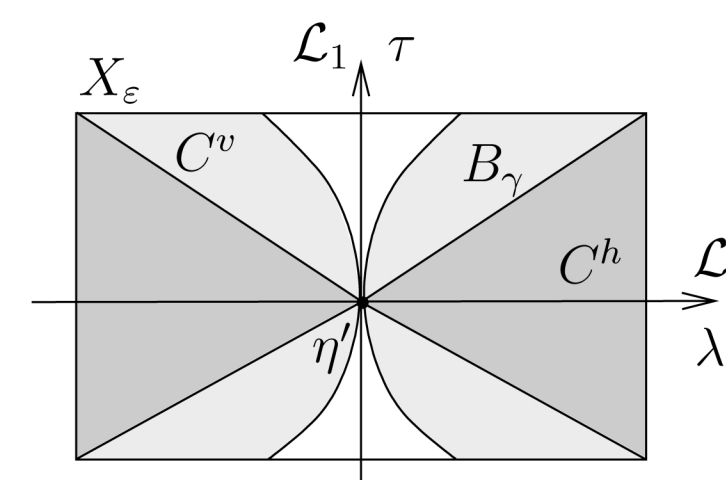
Let $\Omega_n := \{x : f^{-n}\{x\} \subset \Omega_0\}$ and C_n be the critical set for \mathcal{R}^n . Note that \mathcal{R} has topological degree 8. For $x \in \Omega_n \setminus C_n$, define

$$\varphi(x) = \frac{1}{8^n} \sum_{i=1}^{8^n} \varphi(y_i)^{4^n},$$

where $\{y_i\} = \mathcal{R}^{-n}\{x\}$. This average over the fibers avoids monodromy. For all $x \in \Omega_n \setminus C_n$, φ is holomorphic on each branch of f^{-n} by the Inverse Function Theorem. Since φ is bounded on Ω_0 , it is bounded on Ω_n , so by the Riemann Extension Theorem, φ is defined and holomorphic on Ω_n . Since S has local product structure, it is isolated in the recurrent set, so orbits enter and leave Ω_0 at most once. Thus, both definitions of φ in $\Omega_n \cap \Omega_m$ for $n \neq m$ agree. Define $\Omega_\infty = \bigcup_{n=0}^{\infty} \Omega_n$, and define Ω to be the connected component of $\Omega_\infty \cap f^{-1}(\Omega_\infty)$ containing S .

Domination Near η'

Let $X_\varepsilon := \{|\lambda| < \varepsilon, |\tau| < \varepsilon\}$, $C^v := \{|\tau| \geq |\lambda|\}$, and $C^h := \{|\tau| \leq |\lambda|\}$. Consider the "bullet-shaped" regions $B_\gamma := \{(\lambda, \tau) : |\lambda| \geq |\tau|^\gamma\}$.



Proposition 8: If $\varepsilon > 0$ is sufficiently small, then for any $\gamma \in \mathbb{Z}_+$, if $(\lambda_0, \tau_0) \in X_\varepsilon \setminus \mathcal{L}_0$, then $|\lambda_n|/|\tau_n|^\gamma \rightarrow 0$.

This implies that any point near η' and not on \mathcal{L}_0 converges to η' with an arbitrarily high degree of tangency to \mathcal{L}_1 . This is a result of the fact that $\mathcal{R}|_{\mathcal{L}_0} : \lambda \mapsto \lambda^4$ and \mathcal{L}_0 is transversally superattracting like $\tau \mapsto \tau^2$.

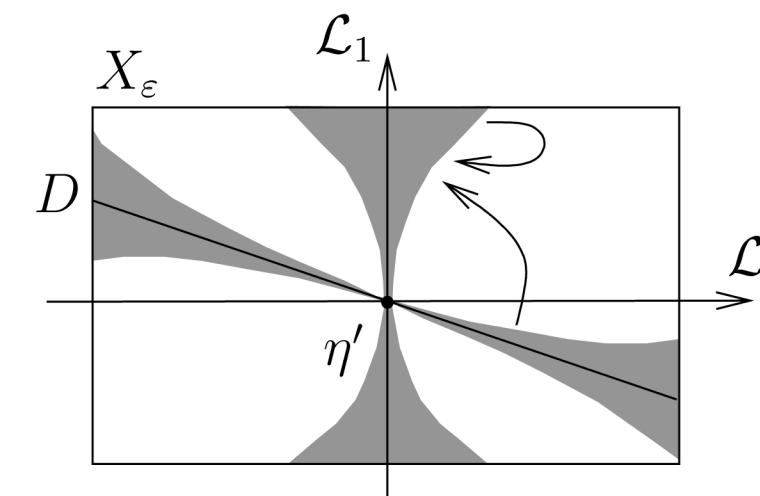
Corollary 9: For any $\gamma \in \mathbb{Z}_+$, $\bigcap_{n=0}^{\infty} \mathcal{R}^{-n}(B_\gamma) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon$

Proposition 10: For $\varepsilon > 0$ sufficiently small and any $\gamma \geq 1$,

- If $(\lambda_i, \tau_i) \in B_\gamma \cap X_\varepsilon$ for $i = 0, \dots, n$, then $|\lambda_n| \asymp |\lambda_0|^{4^n}$.
- If $(\lambda_i, \tau_i) \in X_\varepsilon \setminus B_\gamma$ for $i = 0, \dots, n$, then $|\tau_n| \asymp |\tau_0|^{2^n}$.

Theorem 2: Sketch of Proof Cont.

Let $D := \{\lambda + 2\tau = 0\}$. Note that $D, \mathcal{L}_1 \subset \mathcal{R}^{-1}\{\mathcal{L}_1\}$.

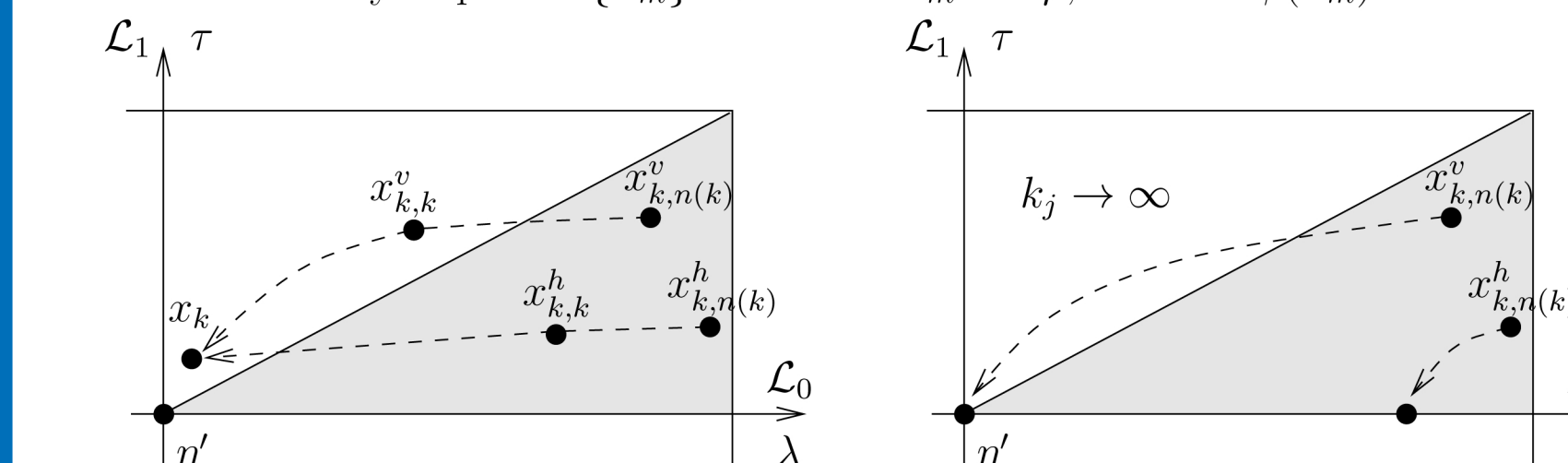


Lemma 11: If $x \in X_\varepsilon \setminus B_\gamma$ and $\varepsilon > 0$ is sufficiently small, then

$$\mathcal{R}^{-1}\{x\} \cap C^h \neq \emptyset \text{ and } \mathcal{R}^{-1}\{x\} \cap C^v \neq \emptyset.$$

Proposition 12: For any $\varepsilon > 0$ sufficiently small, there is a sequence $\{x_k\}$ converging to η' such that for each k , x_k has a preorbit of length k contained in $C^v \cap X_\varepsilon$ and a preorbit of length k contained in $C^h \cap X_\varepsilon$. Moreover, any preimage of x_k that is in X_ε is in Ω .

Let $\{x_{k,i}^v\}$ and $\{x_{k,i}^h\}$ be the preorbits remaining in C^v and C^h . For each k , there is an $n(k)$ such that $x_{k,n(k)}^h$ is the last preimage of x_k remaining in $X_\varepsilon \cap C^h$. Using the invariance $\phi(\mathcal{R}^n(x)) = \phi(x)^{4^n}$, this implies $|\phi(x_{k,n(k)}^v)| = |\phi(x_{k,n(k)}^h)|$ for every $n(k)$. It can also be shown that for any sequence $\{x_m\} \subset \Omega$ with $x_m \rightarrow \eta'$, we have $\phi(x_m) \rightarrow 0$.



Now vary k . By distortion control,

$$\text{dist}(x_{k,n(k)}^v, \eta') \leq A \text{dist}(x_k, \eta')^{\frac{1}{2^{k-n(k)}}} \asymp A \text{dist}(x_{k,n(k)}^h, \eta')^{\frac{n(k)}{2^{k-n(k)}}} \leq A\varepsilon^{2^k}$$

Thus, the sequence $x_{k,n(k)}^v$ converges to η' , so $\phi(x_{k,n(k)}^v)$ converges to 0. Also,

$$x_{k,n(k)}^h \in \bigcap_{i=0}^{n(k)-1} \mathcal{R}^{-i}(C^h) \cap X_\varepsilon$$

For any k , by construction, $x_{k,n(k)}^h \in X_\varepsilon \cap \mathcal{R}(X_\varepsilon)$. Thus, there exists k_j such that $\{x_{k_j, n(k_j)}^h\} \rightarrow x_* \in \mathcal{L}_0 \setminus \{\eta', \eta\}$, so there is a subsequence k_j such that

$$x_{k_j, n(k_j)}^v \rightarrow \eta' \text{ and } x_{k_j, n(k_j)}^h \rightarrow x_* \in \mathcal{L}_0 \setminus \{\eta', \eta\}.$$

Thus, $\phi(x_*)$ is bounded away from 0, a contradiction.

This contradiction implies there is an $x_0 \in \mathcal{B}$ at which $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic. A result by Baouendi and Rothschild implies that if $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at x , then $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $\mathcal{R}(x)$. If $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic in a neighborhood U of $x \in \mathcal{B}$, then since \mathcal{R} is expanding on \mathcal{B} , $\mathcal{R}^n(U)$ covers \mathcal{B} . Then $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at x_0 , where $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ was supposed to be not real analytic. If $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $x \notin \mathcal{B}$, then we may use the Levi foliation to show that $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is real analytic at $\phi(x) \in \mathcal{B}$. Therefore, $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ is not real analytic at any point.

Physical Interpretation

Theorem 2 is related to the Ising Model on the diamond Hierarchical Lattice (DHL). We refer the reader to BLR for physical background. The DHL is a sequence of graphs Γ_n obtained in a self-similar way. Associated to each graph is a partition function $\mathbf{Z}_n(z, t)$ whose zeros

$$\mathcal{S}_n^c := \{(z, t) \in \mathbb{C}^2 : \mathbf{Z}_n(z, t) = 0\}$$

describe the singularities of the Ising model associated to Γ_n . They are called the *Lee-Yang-Fisher zeros*. The actual physics is described by the limit $n \rightarrow \infty$. It is proved in BLR that the limiting distribution of zeros exists as a closed, positive $(1, 1)$ -current \mathcal{S}^c on \mathbb{P}^2 . In fact, $\mathcal{S}^c = \frac{1}{2}\Psi^*S$, where S is the Green current for R . The support of \mathcal{S}^c describes locus where phase transitions occur in \mathbb{C}^2 .

It is shown in BLR that at low complex temperatures $\text{supp } \mathcal{S}^c$ coincides with $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$. Combining Theorem 2 with the work from BLR gives the following:

Corollary 13: At low complex temperatures, the locus of phase transitions for the Ising model on the DHL is a 3 real-dimensional manifold that is C^∞ but not real analytic.

Open Questions

Theorem 1 naturally leads one to question whether there are necessary conditions for $\mathcal{W}_{\text{loc}}^s(S)$ to be real analytic. If $a < b$, the superattracting direction is not as strong as the expansion within L , and I believe there should be a way to exploit this.

Conjecture: For a generic class of mappings with $a < b$, $\mathcal{W}_{\text{loc}}^s(S)$ is not real analytic.

Here I am trying to generalize of the technique in the proof of Theorem