EXAMPLES OF RATIONAL MAPS OF $\mathbb{CP}^2$ WITH EQUAL DYNAMICAL DEGREES AND NO INVARIANT FOLIATION

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Abstract. We present simple examples of rational maps of the complex projective plane with equal first and second dynamical degrees and no invariant foliation.

1. Introduction

A meromorphic map $\varphi : X \dashrightarrow X$ of a compact Kähler manifold X induces a well-defined pullback action $\varphi^* : H^{p,p}(X) \to H^{p,p}(X)$ for each $1 \leq p \leq \dim(X)$. The p-th dynamical degree

$$\lambda_p(\varphi) := \lim_{n\to\infty} \|(\varphi^n)^* : H^{p,p}(X) \to H^{p,p}(X)\|^{1/n} \tag{1}$$

describes the asymptotic growth rate of the action of iterates of φ on $H^{p,p}(X)$. Originally, the dynamical degrees were introduced by Friedland [21] and later by Russakovskii and Shiffman [30] and shown to be invariant under birational conjugacy by Dinh and Sibony [18]. Note that dynamical degrees were originally defined with the limit in (1) replaced by a limsup. However, it was shown in [18, 17] that the limit always exists.

One says that φ is *cohomologically hyperbolic* if one of the dynamical degrees is strictly larger than all of the others. In this case, there is a conjecture [23, 25] that describes the ergodic properties of φ. This conjecture has been proved in several particular sub-cases [24, 13, 15].

What happens in the non-cohomologically hyperbolic case? If a meromorphic map preserves a fibration, then there are nice formulae relating the dynamical degrees of the map [14, 16]. This is the case in the following two examples:

a) It was shown in [12] that bimeromorphic maps of surfaces that are not cohomologically hyperbolic ($\lambda_1(\varphi) = \lambda_2(\varphi) = 1$) always preserve an invariant fibration.
b) Meromorphic maps that are not cohomologically hyperbolic arise naturally when studying the spectral theory of operators on self-similar spaces [31, 1, 22, 2]. All the examples studied in that context preserve an invariant fibration. In several cases, this fibration made it significantly easier to compute the limiting spectrum of the action.

Based on this evidence, Guedj asked in [25, p. 103] whether every non-cohomologically hyperbolic map preserves a fibration.

We will prove:

Theorem 1.1. *The rational map $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ given by*

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X - Z) : -(X + Z)(X - Z)] \tag{2}$$

is not cohomologically hyperbolic ($\lambda_1(\varphi) = \lambda_2(\varphi) = 2$), and no iterate of φ preserves a singular holomorphic foliation. Moreover, for a Baire generic set of automorphisms $A \in \mathrm{PGL}(3, \mathbb{C})$, the composition $A \circ \varphi^4$ has the same properties.

Since preservation of a fibration is a stronger condition than preservation of a singular foliation, φ provides an answer to the question posed by Guedj.

After reading a preliminary version of this paper, Charles Favre asked is the same behavior can be found for a polynomial map of $\mathbb{C}^2$. In [19, §7.2], there is a list of seven types of non-cohomologically hyperbolic polynomial mappings. Our method of proof does not apply in most of these examples for trivial reasons, but we found a map of type (3) for which the same result holds:

Theorem 1.2. *The polynomial map* $\psi(x,y) := \big(x(x-y)+2, (x+y)(x-y)+1\big)$ *extends as a rational map* $\psi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ *that is not cohomologically hyperbolic (*$\lambda_1(\psi) = \lambda_2(\psi) = 2$*), and no iterate of* ψ *preserves a singular holomorphic foliation on* $\mathbb{CP}^2$*.*

Non-cohomologically hyperbolic maps of 3-dimensional manifolds X arise naturally as certain pseudo-automorphisms that are "reversible" on $H^{1,1}(X)$ [5, 4, 29]. For these mappings it follows from Poincaré duality that $\lambda_1(\varphi) = \lambda_2(\varphi)$. Recently, Bedford, Cantat, and Kim [3] have found a reversible pseudo-automorphism of an iterated blow-up of $\mathbb{P}^3$ which does not preserve any invariant foliation. It also answers the question posed by Guedj.

Many authors have studied meromorphic (and rational) maps that preserve foliations and algebraic webs, including [20, 7, 28, 10, 11, 6, 12]. Since the proof of Theorem 1.1 is self-contained and provides insight on the mechanism that prevents φ from preserving a foliation, we will provide a direct proof rather than appealing to results from these previous papers.

Let us give a brief idea of the proof of Theorem 1.1. The fourth iterate φ^4 has an indeterminate point p that is blown-up by φ^4 to a singular curve C. Any foliation $\mathcal{F}$ must be either generically transverse to C or have C as a leaf. This allows us to show that $(\varphi^4)^*\mathcal{F}$ must be singular at either at p or at some preimage r of the singular point. Both of these points have infinite φ^4 pre-orbits, at each point of which φ^{4n} is a finite map. This generates a sequence of distinct points $\{a_{-n}\}_{n=0}^{\infty}$ at such that $(\varphi^{4n})^*\mathcal{F}$ is singular at a_{-n}. If $(\varphi^\ell)^*\mathcal{F} = \mathcal{F}$ for some ℓ this implies that $\mathcal{F}$ is singular at infinitely many points, providing a contradiction. The same method of proof applies for the polynomial map in Theorem 1.2.

Question. *Our proof of Theorem 1.2 relies on the compactness of* $\mathbb{CP}^2$*. Does* ψ *preserve a foliation on* $\mathbb{C}^2$*?*

It would also be interesting to place the maps φ, ψ in the context of the conjecture from [23]:

Question. *Do these maps have topological entropy* $\log 2$*? Can one find a (unique) measure of maximal entropy? As* $n \to \infty$*, what is the asymptotic behavior of the periodic points of period* n*, and are the periodic points predominantly saddle-type, repelling, or degenerate?*

In §2 we provide background on the transformation of foliations and fibrations by rational maps. In §3 we describe some basic properties of φ and we prove that $\lambda_1(\varphi) = 2 = \lambda_2(\varphi)$, showing that φ is not cohomologically hyperbolic. In §4 we prove that no iterate of φ preserves a foliation, and conclude the section with a summary of the properties of φ that make the proof of Theorem 1.1 work. In §5 we prove Theorem 1.2 by computing that $\lambda_1(\psi) = 2 = \lambda_2(\psi)$ and verifying that ψ has the properties listed in §4. The last section, §6, deals with the case of generic rotations of φ, completing the proof of Theorem 1.1.

2. Background

In this section we will present some background about singular holomorphic foliations on complex surfaces and their pullback under rational maps. More details can be found in [27, 8, 6, 20].

The *standard holomorphic foliation on* $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

Let X be a (potentially non-compact) complex surface. A *holomorphic foliation* $\mathcal{F}$ on X is the partition

$$X = \bigsqcup_{\alpha} L_\alpha$$

of X into the disjoint union of connected subsets L_α, which is locally biholomorphically equivalent to the standard holomorphic foliation on $\mathbb{D}^2$. Each L_α is a Riemann surface called a *leaf* of $\mathcal{F}$.

A *singular holomorphic foliation* on X is a foliation $\mathcal{F}$ on $X \setminus P$, where P is a discrete set of points through which $\mathcal{F}$ does not extend. The points of P are called the *singular points* of $\mathcal{F}$. We will abuse notation and denote the corresponding singular foliation by $\mathcal{F}$, as well.

It is easy to write $\mathcal{F}$ as the integral curves of some holomorphic vector field in a neighborhood of any $x \in X \setminus P$. This can also be done in a neighborhood of any singular point:

Theorem 2.1 (Ilyashenko [26] and [27, Thm. 2.22])**.** *In a neighborhood of each singular point $p \in P$, $\mathcal{F}$ is generated as the integral curves of some holomorphic vector field.*

For the remainder of the article we will use the word "foliation" to mean singular holomorphic foliation. Theorem 2.1 allows for any foliation $\mathcal{F}$ on a surface X to be described in the following two equivalent ways:

(i) By an open cover $\{U_i\}$ of X and a system of holomorphic vector fields v_i on U_i with isolated zeros that satisfy the compatibility condition that $v_i = g_{ij} v_j$ for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

(ii) By an open cover $\{U_i\}$ of X and a system of holomorphic one-forms ω_i on U_i with isolated zeros that satisfy the compatibility condition that $\omega_i = g_{ij}\omega_j$ for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

If $\{(U_i, v_i)\}$ satisfies the compatibility condition in (i) we will call it a *compatible system of vector fields*, and if $\{(U_i, \omega_i)\}$ satisfies the compatibility condition in (ii) we will call it a *compatible system of one-forms*. Within a given U_i the tangent direction to a leaf is described by either v_i or the kernel of ω_i. The zeros of v_i or ω_i, correspond to the singular points of $\mathcal{F}$.

Lemma 2.2. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are foliations on a surface X that are equal outside of some analytic curve C. Then, $\mathcal{F} = \mathcal{G}$.*

Proof. It suffices to prove the statement locally, so we can suppose that X is an open subset of $\mathbb{C}^2$. The tangent lines to leaves of $\mathcal{F}$ define a meromorphic function $f : X \dashrightarrow \mathbb{P}^1$, whose indeterminate points are precisely the singular points P of $\mathcal{F}$. Similarly, $\mathcal{G}$ defines a meromorphic function $g : X \dashrightarrow \mathbb{P}^1$, whose indeterminate points are precisely the singular points Q of $\mathcal{G}$. Both f and g are holomorphic on $X \setminus (P \cup Q)$ and equal on the connected set $X \setminus (P \cup Q \cup C)$. Thus, by uniqueness properties of analytic functions they are equal on $X \setminus (P \cup Q)$. Moreover, $P = Q$, since otherwise, one of the functions would define an extension of the other function through a point of its indeterminacy. $\square$

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$. The *pullback* $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \hat{\omega}_i)\}$ where each form $\hat{\omega}_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function in order to eliminate any non-isolated zeros.

Lemma 2.3. *Let $\varphi : X \to Y$ be a dominant holomorphic map between complex surfaces that is finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.*

Proof. Let U be a neighborhood of $y := \varphi(x)$ chosen so that $\mathcal{F}$ is defined on U by a one-form ω vanishing only at y. Since φ is finite at x, there is some neighborhood $V \subset \varphi^{-1}(U)$ of x so that $x = \varphi^{-1}(y) \cap V$. Thus, x is the only zero of $\varphi^*\omega$ within V, so that $\varphi^*\omega$ does not need to be rescaled to define $\varphi^*\mathcal{F}$ in a neighborhood of x. We conclude that x is singular for $\varphi^*\mathcal{F}$. $\square$

For the remainder of the paper, we will restrict our attention to complex projective algebraic surfaces, so that we can discuss rational maps. We will refer to them simply as "surfaces". Let us make the convention that all rational maps are *dominant*, meaning that the image is not contained within a proper subvariety of the codomain.

Let $\varphi : X \dashrightarrow Y$ be a rational map of surfaces and let p be a point from the indeterminacy set $\mathcal{I}_\varphi$. A sequence of point blow-ups $\pi : \widetilde{X} \to X$ *resolves the indeterminacy at* p if φ lifts to a rational map $\widetilde{\varphi} : \widetilde{X} \dashrightarrow X$ which is holomorphic in a open neighborhood of $\pi^{-1}(p)$ and makes the following diagram commute

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"] \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ X \arrow[r, dashed, "\varphi"] & Y, \end{tikzcd} \tag{3}$$

wherever $\widetilde{\varphi}$ and $\varphi \circ \pi$ are both defined.

It is a well-known fact that one can do a sequence of point blow-ups $\pi : \widetilde{X} \to X$ over $\mathcal{I}_\varphi$ so that φ lifts to a holomorphic map $\widetilde{\varphi} : \widetilde{X} \to X$ resolving all of the indeterminate points of $\mathcal{I}_\varphi$. See, for example, [32, Ch. IV, §3.3].

Definition 1. Let $\varphi : X \dashrightarrow Y$ be a dominant rational map and let $\mathcal{F}$ be a foliation on Y. The *pullback of* $\mathcal{F}$ *under* φ is defined by $\varphi^*\mathcal{F} := \pi_*\widetilde{\varphi}^*\mathcal{F}$, where $\pi : \widetilde{X} \to X$ is a sequence of blow-ups over $\mathcal{I}_\varphi$ and $\widetilde{\varphi} : \widetilde{X} \to X$ is a holomorphic map resolving all of the indeterminacy of φ.

The push-forward π_* is defined as follows. The inverse $\pi^{-1} : X \dashrightarrow \widetilde{X}$ is a birational map whose restriction $\pi^{-1}|_{X \setminus \mathcal{I}_\varphi}$ is a biholomorphism onto its image. Any foliation $\mathcal{G}$ on $\widetilde{X}$ has finitely many singular points since $\widetilde{X}$ is compact. Thus, $\left(\pi^{-1}|_{X\setminus\mathcal{I}_\varphi}\right)^* \mathcal{G}$ is a foliation on $X \setminus \mathcal{I}_\varphi$ having finitely many singular points. Since $\mathcal{I}_\varphi$ is also a finite set, the result is a singular holomorphic foliation on X. More specifically, at any $x \in \mathcal{I}_\varphi$, the pull-back $\left(\pi^{-1}|_{X\setminus\mathcal{I}_\varphi}\right)^* \mathcal{G}$ will either extend through x as a regular holomorphic foliation or, by definition, x will be a singular point. The result is called $\pi_*\mathcal{G}$. Using Theorem 2.1, one can re-express $\pi_*\mathcal{G}$ using a compatible system of vector fields $\{(U_i, v_i)\}$ or a compatible system of one-forms $\{(U_i, \omega_i)\}$.

Remark. The definition of $\varphi^*\mathcal{F}$ given above coincides with the foliation obtained by extending $\left(\varphi|_{X\setminus\mathcal{I}_\varphi}\right)^* \mathcal{F}$ through $\mathcal{I}_\varphi$, so it is independent of the choice of blow-up. The blow-up is used to make it clear that the singular points of $\left(\varphi|_{X\setminus\mathcal{I}_\varphi}\right)^* \mathcal{F}$ do not accumulate on $\mathcal{I}_\varphi$.

We will use the following three well-known lemmas.

Lemma 2.4. *Let* $\varphi : X \dashrightarrow Y$ *and* $\psi : Y \dashrightarrow Z$ *be any dominant rational maps. For any foliation* $\mathcal{F}$ *on* Z *we have* $(\psi \circ \varphi)^*\mathcal{F} = \varphi^*(\psi^*\mathcal{F})$.

Proof. The foliations $(\psi \circ \varphi)^*\mathcal{F}$ and $\varphi^*(\psi^*\mathcal{F})$ agree on $X \setminus (\mathcal{I}_\varphi \cup \varphi^{-1}(\mathcal{I}_\psi))$. Since $\varphi^{-1}(\mathcal{I}_\psi)$ is at most an analytic curve, the result then follows from Lemma 2.2. ☐

Lemma 2.5. *Let* $\mathcal{F}$ *be a foliation on a surface* X *and suppose* $C \subset X$ *is an irreducible algebraic curve. Then, either* C *is a leaf of* $\mathcal{F}$ *or* C *is transverse to* $\mathcal{F}$ *away from finitely many points.*

Proof. Let $\psi : \widetilde{C} \to X$ be the normalization of C. (For background on normalization, see [32, Ch. II, §5].) Since C is irreducible, $\widetilde{C}$ is a connected Riemann surface. Let $\{(U_i, \omega_i)\}$ be a compatible system of one-forms describing $\mathcal{F}$. Then, the zeros of $\{(\psi^{-1}(U_i), \psi^*\omega_i)\}$ are a well-defined analytic subset of $\widetilde{C}$. Any point of $C \setminus \mathrm{Sing}(C)$ where C is parallel to $\mathcal{F}$ corresponds to the image under ψ of such a zero, so the the result follows. ☐

Lemma 2.6. *Let* $\varphi : X \dashrightarrow Y$ *be a dominant rational map of surfaces, let* $p \in \mathcal{I}_\varphi$, *and suppose* $\pi : \widetilde{X} \to X$ *and* $\widetilde{\varphi} : \widetilde{X} \dashrightarrow Y$ *give a resolution of the indeterminacy at* p.

If $\mathcal{F}$ *is a foliation on* Y *and one of the irreducible components of* $\widetilde{\varphi}(\pi^{-1}(p))$ *is generically transverse to* $\mathcal{F}$, *then* $\varphi^*\mathcal{F}$ *is singular at* p.

Proof. Let C be an irreducible component of $\widetilde{\varphi}(\pi^{-1}(p))$ that is generically transverse to $\mathcal{F}$ and suppose $\mathcal{F}$ is represented by a compatible system of one-forms $\{(U_i, \omega_i)\}$. Then, the tangent spaces to generic points of C are not in the kernels of any of the one-forms based at these points.

Let $E \subset \pi^{-1}(p)$ be an irreducible component that is mapped by $\widetilde{\varphi}$ onto C. Then, at generic $x \in E$ we have that x and $\widetilde{\varphi}(x)$ are smooth points of E and C, respectively, and $D\widetilde{\varphi}$ is an isomorphism between their tangent spaces. This implies that $\{(\widetilde{\varphi}^{-1}(U_i), \widetilde{\varphi}^*\omega_i)\}$ are non-vanishing at generic points of E and have kernel transverse to E at these points. Thus, $\widetilde{\varphi}^*\mathcal{F}$ is generically transverse to E.

Now we can choose two disjoint holomorphic disks γ_1 and γ_2 within leaves of $\widetilde{\varphi}^*\mathcal{F}$ that intersect $\pi^{-1}(p)$ transversally at two distinct points of $x_1, x_2 \in E$. Then, $\pi(\gamma_1)$ and $\pi(\gamma_2)$ are distinct integral curves of $\varphi^*\mathcal{F} = \pi_*\widetilde{\varphi}^*\mathcal{F}$ going through p. This implies that p is a singular point for $\varphi^*\mathcal{F}$. □

A *fibration* is a dominant rational map $\rho : X \dashrightarrow S$ to a non-singular algebraic curve S with the property that each fiber is connected. The level curves of ρ define a foliation on $X \setminus \mathcal{I}_\rho$ having finitely many singularities. It thus extends as a singular foliation $\mathcal{F}$ on X called the *foliation induced by* ρ.

Lemma 2.7. *Let $\rho : X \dashrightarrow S$ be a fibration inducing a foliation $\mathcal{F}$ on X. A rational map $\varphi : X \dashrightarrow X$ preserves $\mathcal{F}$ if and only if ρ semi-conjugates φ to a holomorphic map of S:*

(4)
$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\rho"] & X \arrow[d, dashed, "\rho"] \\ S \arrow[r, "\eta"] & S. \end{tikzcd}$$

Proof. Suppose that ρ semi-conjugates φ to a holomorphic map $\eta : S \to S$. We will show that $\varphi^*\mathcal{F} = \mathcal{F}$. By Lemma 2.2, it suffices to prove equality in the complement of a finite set of curves and points.

If $\{(U_i, \psi_i)\}$ is any system of coordinates on S, then $\mathcal{F}$ is defined on $X \setminus \mathcal{I}_\rho$ by the compatible system of one-forms $\{(\rho^{-1}(U_i), \mathrm{d}(\psi_i \circ \rho))\}$. For any $x \in X \setminus (\rho^{-1}(\mathrm{crit}(\eta)) \cup \varphi^{-1}\mathcal{I}_\rho \cup \mathcal{I}_\rho \cup \mathcal{I}_\varphi)$ we can express $\mathcal{F}$ by $\mathrm{d}(\psi_i \circ \eta \circ \rho)$, where ψ_i is some chart defined in the neighborhood of $\eta \circ \rho(x)$, since then $\psi_i \circ \eta$ serves as a chart in the neighborhood of $\rho(x)$. Meanwhile, commutativity of Diagram (4) implies that in a neighborhood of x we have

$$\varphi^*\mathrm{d}(\psi_i \circ \rho) = \mathrm{d}(\psi_i \circ \rho \circ \varphi) = \mathrm{d}(\psi_i \circ \eta \circ \rho).$$

Thus, $\mathcal{F}$ and $\varphi^*\mathcal{F}$ agree in a neighborhood of x.

Now suppose that $\varphi : X \dashrightarrow X$ preserves the foliation $\mathcal{F}$. It is possible that φ collapses $\rho^{-1}(S_0)$ into $\mathcal{I}_\rho$ for some finite set $S_0 \subset S$. For any $s \in S \setminus S_0$ we define $\eta(s) = \rho(\varphi(x))$ where x is any element of $\rho^{-1}(s) \setminus (\mathcal{I}_\varphi \cup \varphi^{-1}(\mathcal{I}_\rho))$. Since the fibers of ρ are connected, φ maps fibers to fibers. Therefore, this results in a well-defined function $\eta : S \setminus S_0 \to S$ which makes (4) commute wherever $\eta \circ \rho$ and $\rho \circ \varphi$ are both defined.

For any $s \in S$ there is a holomorphic disc $\mathcal{D}$ in X intersecting $\rho^{-1}(s)$ transversally, that is small enough so that $\rho : \mathcal{D} \to \rho(\mathcal{D}) \subset S$ is one-to-one. (We are again using that each fiber is connected.) Let $\chi : \rho(\mathcal{D}) \to \mathcal{D}$ be its inverse. Then, $\rho \circ \varphi \circ \chi : \rho(\mathcal{D}) \to S$ is a meromorphic (hence holomorphic) function that agrees with η on $\rho(\mathcal{D}) \setminus S_0$. We conclude that η extends as a holomorphic map $\eta : S \to S$.

□

Remark. Using Lemma 2.7, it follows immediately from Theorem 1.1 that no iterate of φ preserves a fibration.

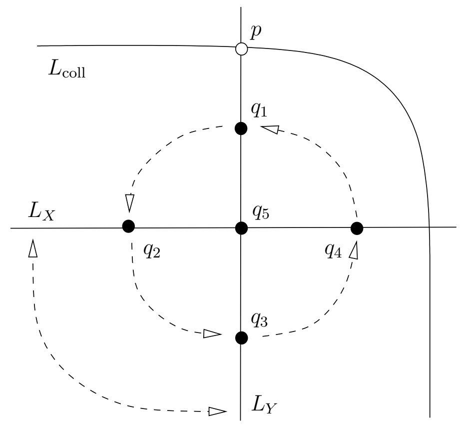

FIGURE 1. Postcritical set for φ

3. STRUCTURE OF φ

Recall that $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

3.1. **Indeterminate and Postcritical Sets.** Solving for the points where all three homogeneous coordinates of φ are zero, we find that the only indeterminate point of φ is

$$p := [1 : 0 : 1]. \tag{5}$$

We have $|D\varphi| = -4Y(X-Z)^2$, so the curves

$$L_{\mathrm{coll}} \;:=\; \{X = Z\} \text{ and} \tag{6}$$

$$L_Y \;:=\; \{Y = 0\} \tag{7}$$

are critical. Let $L_X := \{X = 0\}$, and observe that

$$\begin{tikzcd} L_{\mathrm{coll}} \setminus \{p\} \arrow[r] & {[1:0:0]} \arrow[r] & {[0:1:-1]} \arrow[r] & {[-1:0:1]} \arrow[r] & {[0:1:0]} \arrow[lll, bend left] \end{tikzcd} \tag{8}$$

We will refer to the points in this four cycle as $q_1, \ldots, q_4$, respectively. Meanwhile,

$$\begin{tikzcd} L_Y \arrow[r] & L_X \arrow[l, bend left] \end{tikzcd}$$

as illustrated in Figure 1.

3.2. **Resonant Dynamical Degrees.** Note that the orbit of the collapsed curve L_{coll} lies in the four cycle (8), which is disjoint from p. It follows that there is no curve V such that $\varphi(V) \subset \mathcal{I}_\varphi$, so φ is algebraically stable by [33, Prop. 1.4.3]. Therefore, $\lambda_1(\varphi) = \deg_{\mathrm{alg}}(\varphi) = 2$.

We will show in the proof of Lemma 3.2 that p can be resolved by two blow-ups with the image of the exceptional divisors being the line $\{Y = -2Z\}$. In particular, $[1 : 1 : -1]$ is neither a critical value nor in the image of the indeterminacy, so it is a generic point for φ. Its two preimages under φ are $[1 : \pm i : 0]$, so $\lambda_2(\varphi) = 2$. This establishes the following resonance of dynamical degrees:

Lemma 3.1. $\lambda_1(\varphi) = \lambda_2(\varphi) = 2$ *so that φ is not cohomologically hyperbolic.*

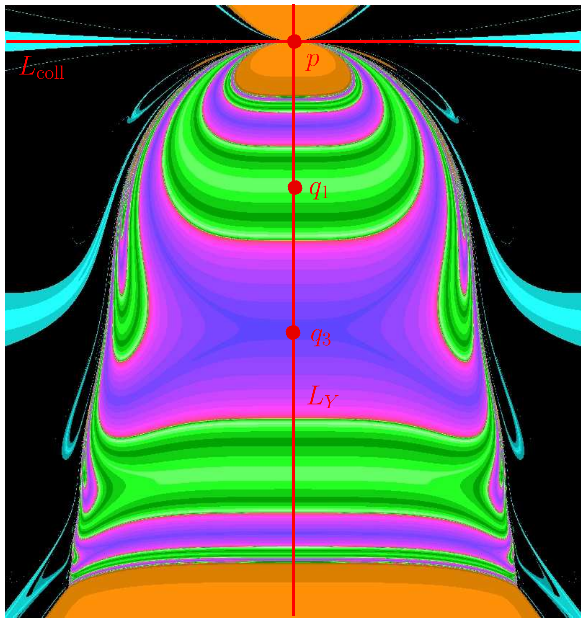

FIGURE 2. Real slice in local coordinates (y, z)

3.3. **Fatou Set.** Note also that φ^2 fixes both L_Y and L_X, with each line transversally superattracting under φ^2. It is easy to compute that $\varphi^2 \mid L_Y$ is conjugate to $z \mapsto z^2 - 1$, the well-known Basilica map. The period two superattracting cycle is $q_1 \leftrightarrow q_3$. Similarly, $g \mid L_X$ is also conjugate to $z \mapsto z^2 - 1$ and its period two superattracting cycle is $q_2 \leftrightarrow q_4$. Finally, φ (and hence φ^2) has $q_5 = [0 : 0 : 1]$ as a superattracting fixed point.

We obtain five superattracting fixed points for φ^4. Figure 2 shows the real slice in the local coordinates $(y, z) = (Y/X, Z/X)$, with the basins of $q_1, \ldots, q_5$ shown in green, black, purple, blue, and orange, respectively.

Question. *Is the Fatou set of φ the union of the basin of the superattracting 4-cycle $q_1, \ldots, q_4$ and the basin of the superattracting fixed point q_5?*

3.4. **Resolving the indeterminate point** p**.** Proof of Theorem 1.1 relies on careful analysis of φ^4, the fourth iterate of φ. The indeterminacy set of φ^4 is

$$\mathcal{I}_{\varphi^4} = \{p\} \cup \varphi^{-1}(\{p\}) \cup \varphi^{-2}(\{p\}) \cup \varphi^{-3}(\{p\}) =$$
$$\{[1 : 0 : 1]\} \cup \{[0 : \pm i : 1]\} \cup \{[1 : 0 : -1 \pm i]\} \cup \{[0 : i : \pm\sqrt{1 \pm i}]\}.$$

However, we will only need to resolve the indeterminacy of φ^4 at p.

Let $\widehat{\mathbb{P}^2}$ be the blow-up $\mathbb{P}^2$ at p with exceptional divisor E_1, and let $\widehat{L}_{\text{coll}}$ denote the proper transform of L_{coll}. Then let $\widetilde{\mathbb{P}^2}$ be the blow-up of $\widehat{\mathbb{P}^2}$ at the point $E_1 \cap \widehat{L}_{\text{coll}}$, resulting in a new exceptional divisor E_2. We will abuse notation by denoting the proper transform of E_1 in $\widetilde{\mathbb{P}^2}$ also by E_1.

Lemma 3.2. *The map $\varphi^4 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ lifts to a rational map $\widetilde{\varphi^4} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ that resolves the indeterminacy of φ^4 at p. We have that $\widetilde{\varphi^4}(E_1) = [1 : 0 : -9] =: s$ and that $\widetilde{\varphi^4}(E_2)$ is an irreducible algebraic curve C_4 of degree 8 that is singular at s.*

Proof. We will first show that φ lifts to a rational map $\widetilde{\varphi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ that resolves the indeterminacy of φ at p, satisfying $\widetilde{\varphi}(E_1) = [0 : 1 : -2]$ and $\widetilde{\varphi}(E_2) = \{Z = -2Y\}$.

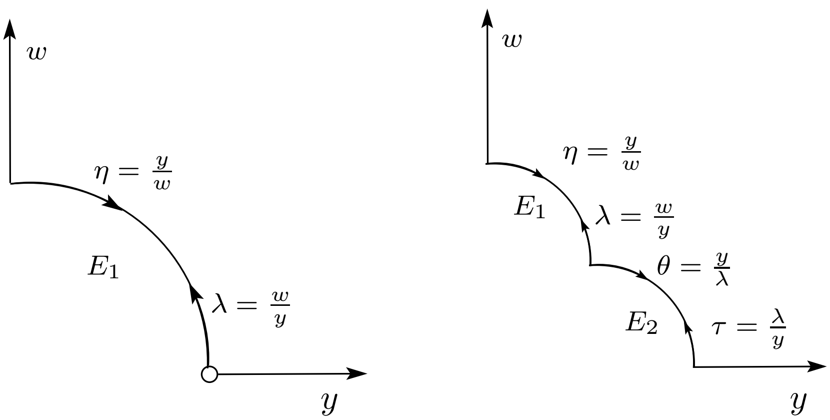

Figure 3. Left: coordinates used for the first blow-up of p. The new indeterminate point $(y, \lambda) = (0, 0)$ for $\widehat{\varphi}$ is marked by an open circle. Right: coordinates used for the second blow-up.

We will do calculations in several systems of local coordinates in a neighborhood of E_1 and E_2. A summary is shown in Figure 3. Consider the two systems of affine coordinates on $\mathbb{P}^2$ given by $(y, z) = (Y/X, Z/X)$ and $(x, \zeta) = (X/Y, Z/Y)$. Let $w := z - 1$, so (y, w) are coordinates that place p locally at the origin. Writing $(x', \zeta') = \varphi(y, w)$, we have

$$(x', \zeta') = \left(\frac{y^2}{w}, -w - 2\right).$$

There are two systems of coordinates in a neighborhood of E_1 within $\widehat{\mathbb{P}^2}$. They are (y, λ), where $w = \lambda y$, and (w, η), where $y = \eta w$. Then, φ lifts to a rational map $\widehat{\varphi} : \widehat{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ which is expressed in these coordinates by

$$(x', \zeta') = \left(\frac{y}{\lambda}, -\lambda y - 2\right) \text{ and } (x', \zeta') = \left(\eta^2 w, -w - 2\right). \tag{9}$$

The exceptional divisor E_1 is given in the first set of coordinates by $y = 0$ and in the second set of coordinates by $w = 0$. It is clear from (9) that $\hat{\varphi}$ extends holomorphically to all points of E_1 other than $\lambda = 0$, sending each of these points to $(x, \zeta) = (0, -2)$.

In the (y, w) coordinates, L_{coll} is given by $w = 0$. Thus, in the (y, λ) coordinates, the proper transform $\widehat{L}_{\text{coll}}$ is given by $\lambda = 0$ so that $E_1 \cap \widehat{L}_{\text{coll}}$ is given by $(0, 0)$, the indeterminate point for $\hat{\varphi}$ on E_1. We now blow this point up using two new systems of coordinates in a neighborhood of the new exceptional divisor E_2. They are (y, τ), where $\lambda = \tau y$, and (λ, θ), where $y = \theta\lambda$.

The map $\widehat{\varphi}$ lifts to a map $\widetilde{\varphi}$, which can be expressed in local coordinates as

$$(y', z') = \left(\tau, -\tau^2 y^2 - 2\tau\right) \text{ and } (x', \zeta') = \left(\theta, -\theta\lambda^2 - 2\right) \tag{10}$$

This shows that $\widetilde{\varphi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ is holomorphic in a neighborhood of E_2. Thus, it is holomorphic in a neighborhood of $E_1 \cup E_2$. Since E_2 is given in these systems of coordinates by $y = 0$ and $\lambda = 0$, respectively, one can see from (10) that $\widetilde{\varphi}(E_2) = \{Z = -2Y\}$.

Notice that the line $C_1 := \{Z = -2Y\}$ passes through no points of

$$\mathcal{I}_{\varphi^3} = \{[1:0:1], [0:\pm i:1], [1:0:-1 \pm i]\}.$$

This implies that

$$\widetilde{\varphi^4} := \varphi^3 \circ \widetilde{\varphi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$$

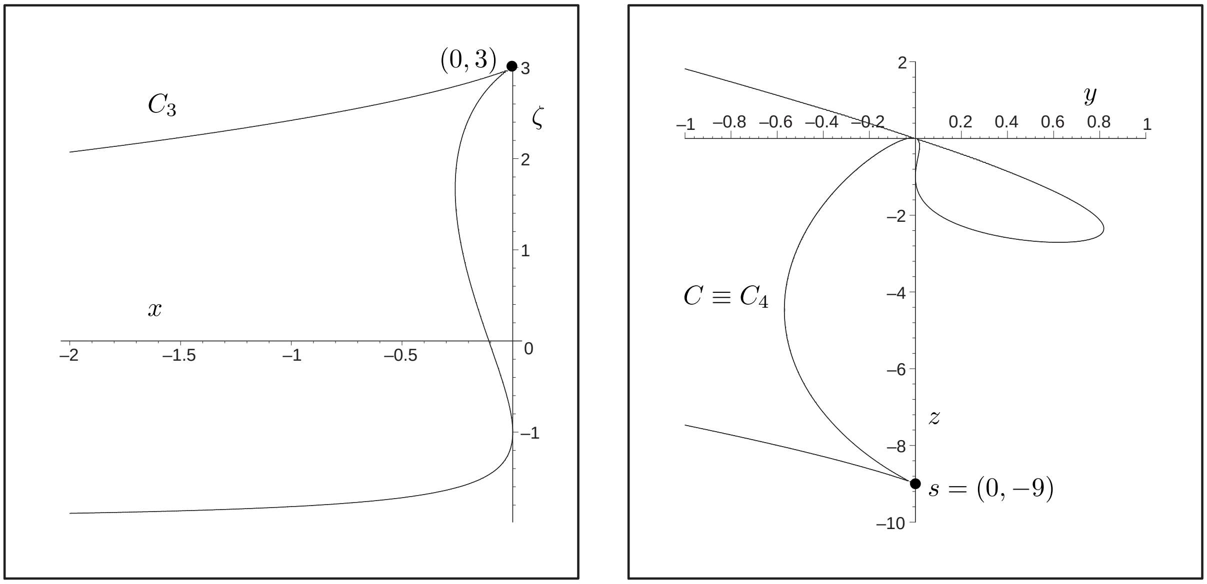

FIGURE 4. The curve C_3 shown on the left in the (x, ζ) coordinates and the curve C_4 shown on the right in the (y, z) coordinates. The cusps at $(x, \zeta) = (0, 3)$ and $(y, z) = (0, -9)$, respectively, are marked.

is holomorphic in a neighborhood of $E_1 \cup E_2$; i.e., that $\widetilde{\varphi^4}$ resolves the indeterminacy of φ^4 at p. Since $\widetilde{\varphi}(E_1) = [0 : 1 : -2]$ it is easy to check that $\widetilde{\varphi^4}(E_1) = s = [1 : 0 : -9] := s$.

Meanwhile, $\widetilde{\varphi}(E_2)$ is the projective line C_1. Its forward image under φ^3 is an algebraic curve $C_4 := \varphi^3(C_1)$ of degree $8 = \deg(\varphi^3)$. Since C_1 does not intersect $\mathcal{I}_{\varphi^3}$, C_4 is irreducible.

It remains to show that C_4 is singular at s. Notice that φ maps a neighborhood of $[0 : 1 : 3]$ biholomorphically to a neighborhood of s. Thus, it will be sufficient to show that $C_3 := \varphi^2(C_1)$ is singular at $[0 : 1 : 3]$. See Figure 4 for plots of the real slices of C_3 and C_4.

Let us work in the (x, ζ) local coordinates. We have

$$(x', \zeta') = \varphi^2(x, \zeta) = \left(\frac{x^2 (x - \zeta)^2}{-1 + x^2 - \zeta^2}, -1 - x^2 + \zeta^2 \right).$$

If we parameterize C_1 by $t \mapsto (t, -2)$ and then compose with φ^2, we obtain the following parameterization of C_3:

$$t \mapsto \left(\frac{t^2 (t + 2)^2}{-5 + t^2}, 3 - t^2 \right). \tag{11}$$

To see that C_3 is singular at $(0, 3)$, it suffices to check that C_3 intersects any line through $(0, 3)$ with intersection number ≥ 2. Such a generic line is given by $Ax + B(\zeta - 3) = 0$. Substituting (11) into this equation, we obtain

$$\frac{At^2 (t + 2)^2}{-5 + t^2} - Bt^2 = 0,$$

which clearly has a double zero at $t = 0$, implying that the intersection number of C_3 with the line given by $Ax + B(\zeta - 3) = 0$ is at least two. $\square$

Remark. Applying φ to the parameterization of C_3 given in (11) one gets a parameterization of C_4. One can then use Gröbner bases [9, Ch. 3, §3] to generate the following implicit equation for of C_4, which is expressed in the local coordinates $(y,u)=(y,z+9)$ centered at s:

$$\begin{aligned} C_4 := \{ & 2624400\,y^2+2099520\,uy+419904\,u^2-1253880\,y^3-2776032\,uy^2-1463832\,u^2y \\ & -291600\,u^3+140049\,y^4+743580\,uy^3+976374\,u^2y^2+416988\,u^3y+84321\,u^4 \\ & +24192\,y^5-12204\,uy^4-145656\,u^2y^3-159984\,u^3y^2-62280\,u^4y-12996\,u^5 \\ & -6912\,y^6-17664\,uy^5-7818\,u^2y^4+12000\,u^3y^3+13108\,u^4y^2+5152\,u^5y \\ & +1126\,u^6+512\,y^7+1792\,uy^6+2176\,u^2y^5+820\,u^3y^4-400\,u^4y^3-496\,u^5y^2 \\ & -224\,u^6y-52\,u^7+u^4y^4+4\,u^5y^3+6\,u^6y^2+4\,u^7y+u^8=0\} \end{aligned} \tag{12}$$

Since the lowest degree terms of (12) are of degree 2, this gives an alternative computational proof that C_4 is singular at s.

4. Proof of non-existence of invariant foliations for φ

Implicitly throughout the proof we assume that $(\varphi^m)^*(\varphi^n)^*\mathcal{F}=(\varphi^{m+n})^*\mathcal{F}$ for $m,n\geq 1$, which follows from Lemma 2.4. Also note that the whole proof will take place in the coordinates $(y,z)=(Y/X,Z/X)$.

The φ-preimage of L_{coll} is the circle $S:=\{y^2+z^2=1\}$ which intersects L_Y at the indeterminate point $p=(0,1)$ and at $q_3=(0,-1)$. It follows that φ^2 is a finite holomorphic map at each point of L_Y other than p and q_3, and consequently, φ^4 is a finite holomorphic map at every point of L_Y except at the finite set of points $\mathrm{NF}:=\{(0,z)\,|\,z=1,-1,0,-2,-1\pm i\}$ consisting of p, q_3, and their φ^2-preimages.

One can check that the point $r:=(0,-1+\sqrt{2}i)\notin \mathrm{NF}$ is a φ^4-preimage of the singular point s of the blow-up curve C. We will first show for any foliation $\mathcal{F}$ that $(\varphi^4)^*\mathcal{F}$ is singular either at the indeterminate point p or at r. By Lemma 2.5, either C is a leaf of $\mathcal{F}$ or generically transverse to $\mathcal{F}$. In the second case, Lemma 2.6 immediately implies that $(\varphi^4)^*\mathcal{F}$ is singular at p. On the other hand, if C is a leaf of $\mathcal{F}$, s must be a singular point for $\mathcal{F}$, since s is a singular point of C by Lemma 3.2. Since φ^4 is a finite holomorphic map at r and $\varphi^4(r)=s$, by Lemma 2.3, $(\varphi^4)^*\mathcal{F}$ is singular at r as well.

Note that neither p nor r are critical points of $\varphi|L_Y$, so they are not exceptional points. One can check that any preorbit of p or r under $\varphi^4|L_Y$ is disjoint from NF. Let a_0 be the point (p or r) where $(\varphi^4)^*\mathcal{F}$ is singular. If we denote some $\varphi^4|L_Y$-preorbit of a_0 by $\{a_{-i}\}_{i=0}^{\infty}$, then for each i, $\left(\varphi^{4(i+1)}\right)^*\mathcal{F}$ will be singular at a_{-i}.

Now suppose the foliation $\mathcal{F}$ is preserved by φ^ℓ for any positive integer ℓ. Then $\mathcal{F}=\left(\varphi^{4\ell k}\right)^*\mathcal{F}$ is singular at $a_{(-\ell k)}$ for all $k\geq 1$. This implies $\mathcal{F}$ has infinitely many singular points, giving a contradiction. □

Observation: It was more convenient to explain the proof with a specific map; however the proof holds in greater generality. In fact, we have established

Proposition 4.1. *Assume the map $\eta:\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfies the following conditions:*

(1) There exists $p\in\mathcal{I}_\eta$ and some iterate k so that η^k blows up p to a singular curve C.
(2) p has an infinite preorbit along which η is a finite holomorphic map.
(3) There is a singular point $s\in C$ having an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

Note that one can replace Conditions (2) and (3) with the equivalent condition that p and s have infinite η^k-preorbits along which η^k is a finite holomorphic map.

5. Proof of non-existence of invariant foliations for ψ

The extension of $\psi(x,y) = \big(x(x-y)+2, (x+y)(x-y)+1\big)$ to $\mathbb{CP}^2$ is

$$\psi[X:Y:Z] = [X(X-Y)+2Z^2 : (X+Y)(X-Y)+Z^2 : Z^2]. \tag{13}$$

Let us first verify that ψ is algebraically stable. The indeterminate set consists of the point

$$p := [1:1:0],$$

and the critical set is

$$\{X=Y\} \cup \{Z=0\}.$$

One can check that $L_{\mathrm{coll}} := \{X=Y\}$ satisfies that $\varphi(L_{\mathrm{coll}} \setminus \{p\}) = [2:1:1]$, which is periodic of period 2:

$$[2:1:1] \leftrightarrow [4:4:1].$$

The line $L_\infty := \{Z=0\}$ is not collapsed by ψ; using the coordinate $w = Y/X$, the restriction $\psi \mid L_\infty$ is given by $w \mapsto w+1$. Since no iterate of a collapsing line lands on the indeterminate point p, it follows from [33, Prop. 1.4.3] that ψ is algebraically stable; thus, $\lambda_1(\psi) = 2$.

Any point $(a,b) \in \mathbb{C}^2$ with $2\,a-b-3 \neq 0$ has two preimages in $\mathbb{C}^2$ given by

$$(x,y) = \left(\pm\frac{-2+a}{\sqrt{2\,a-b-3}}, \mp\frac{a-1-b}{\sqrt{2\,a-b-3}}\right). \tag{14}$$

Since generic points of $\mathbb{C}^2$ do not have a preimage on L_∞, $\lambda_2(\psi) = 2$; thus ψ is not cohomologically hyperbolic.

Proof of Theorem 1.2. It suffices to verify that ψ satisfies the conditions of Proposition 4.1.

A sequence of two blow-ups at p resolves the indeterminacy of ψ. The first blow-up results in a lift $\widehat{\psi} : \widehat{\mathbb{CP}^2} \dashrightarrow \mathbb{CP}^2$ with exceptional divisor E_1. The point $q \in E_1$ with coordinate $\lambda := (Y-X)/Z = 0$ is indeterminate for $\widehat{\psi}$, and $\widehat{\psi}\,(E_1 \setminus \{q\}) = [1:2:0]$. The second blow-up (at q) resolves the indeterminacy, and the lifted map $\widetilde{\psi}$ sends the new exceptional divisor E_2 to the line $C_1 := \{2X-Y-3Z=0\}$. We omit the calculations as they are very similar to those in Lemma 3.2.

The indeterminacy set of ψ^3 is

$$\mathcal{I}_{\psi^3} = \mathcal{I}_\psi \cup \psi^{-1}(\mathcal{I}_\psi) \cup \psi^{-2}(I_\psi) = \{[1:1:0]\} \cup \{[1:0:0]\} \cup \{[1:-1:0]\}.$$

Since $\mathcal{I}_{\psi^3}$ is disjoint from C_1, the map $\widetilde{\psi^4} := \psi^3 \circ \widetilde{\psi}$ resolves the indeterminacy of ψ^4 at p. Moreover, $\widetilde{\psi^4}$ blows up p to the curve

$$C_4 := \widetilde{\psi^4}(E_2) = \psi^3(C_1).$$

We will now check that $s = [4:4:1]$ is a singular point on C_4. Let us work in the affine coordinates $(x,y) = (X/Z, Y/Z)$. If we parameterize C_1 by $t \mapsto (t, 2t-3)$, then a parameterization of $C_3 := \psi^2(C_1)$ is obtained by substituting into the expression for ψ^2:

$$t \mapsto \left(-2\,t^4 + 15\,t^3 - 33\,t^2 + 12\,t + 22\,,\ -8\,t^4 + 66\,t^3 - 187\,t^2 + 204\,t - 59\right). \tag{15}$$

One can see that C_3 is singular at $(2,1)$ by using Gröbner bases to convert (15) into the following implicit equation for C_3

$$\begin{aligned} 256\,u^4 - 256\,u^3v + 96\,u^2v^2 - 16\,uv^3 + v^4 - 5650\,u^3 + 6253\,u^2v & \\ -2228\,uv^2 + 257\,v^3 + 10816\,u^2 - 10816\,uv + 2704\,v^2 &= 0, \end{aligned}$$

which is expressed in local coordinates (u,v) where $x = u+2, y = v+1$. Since the lowest order terms are of degree 2, the point $(x,y) = (2,1)$ is singular for C_3. (This can also be shown using the parameterization of C_3, like in the proof of Lemma 3.2.)

One can check that $D\psi$ is invertible at $(2,1)$ so that $C_4 = \psi(C_3)$ is singular at $s = \psi(2,1) = (4,4)$. Therefore, Condition (1) from Proposition 4.1 holds.

Since the action of ψ on L_∞ is given by $w \mapsto w+1$, and p is given in this coordinate by $w = 1$, we find that p has an infinite pre-orbit under ψ. Moreover, these preimages are disjoint from p (and hence from the collapsing line L_{coll}), so Condition (2) of the proposition holds.

It remains to check Condition (3). Notice that $\psi(L_{\mathrm{coll}} \setminus \{p\}) = (2,1) \in C_1$. Therefore, it suffices to show that $(4,4) \notin C_1$ has an infinite pre-orbit in $\mathbb{C}^2$ consisting of points not on C_1. For any $(a,b) \notin C_1$, the two preimages given by (14) cannot both be on C_1, since in that case

$$\begin{aligned} \frac{-5+3a-b+3\sqrt{2a-b-3}}{\sqrt{2a-b-3}} &= 0, \text{and} \\ \frac{5-3a+b+3\sqrt{2a-b-3}}{\sqrt{2a-b-3}} &= 0. \end{aligned}$$

Summing these equations yields $-6 = 0$, a contradiction. Therefore, any point of $\mathbb{C}^2 \setminus C_1$ has an infinite pre-orbit disjoint from C_1.

Since all three conditions of Proposition 4.1 hold, we conclude that no iterate of ψ preserves a foliation. □

6. Proof of non-existence of invariant foliations for generic rotations of φ^4

Note that for any $A \in \mathrm{PGL}(3,\mathbb{C})$, we have $\mathcal{I}_{A\circ\varphi^4} = \mathcal{I}_{\varphi^4}$. For any $n \in \mathbb{Z}^+$, let

$$\Omega_1^n := \left\{A \in \mathrm{PGL}(3,\mathbb{C}) \,:\, (A\circ\varphi^4)^n(L_{\mathrm{coll}} \setminus \mathcal{I}_{\varphi^4}) \notin \{p,r\}\right\}, \tag{16}$$

$$\Omega_2^n := \left\{A \in \mathrm{PGL}(3,\mathbb{C}) \,:\, p,q \notin (A\circ\varphi^4)^n(\mathcal{I}_{\varphi^4})\right\}, \text{ and} \tag{17}$$

$$\begin{aligned} \Omega_3^n := &\left\{A \in \mathrm{PGL}(3,\mathbb{C}) :\ \text{for all } 0 \le i \ne j \le n,\ (A\circ\varphi^4)^{-i}(p) \cap (A\circ\varphi^4)^{-j}(p) = \emptyset\right\} \cap \\ &\left\{A \in \mathrm{PGL}(3,\mathbb{C}) :\ \text{for all } 0 \le i \ne j \le n,\ (A\circ\varphi^4)^{-i}(r) \cap (A\circ\varphi^4)^{-j}(r) = \emptyset\right\}. \end{aligned} \tag{18}$$

Here, $(\varphi^4)^n(\mathcal{I}_{\varphi^4})$ denotes the total transform of $\mathcal{I}_{\varphi^4}$, i.e. $(\varphi^4)^n(\mathcal{I}_{\varphi^4}) = (\varphi^4)^{n-1}(C_1)$, where φ^4 blows up $\mathcal{I}_{\varphi^4}$ to C_1.

Each of these sets is the complement of an algebraic set because the conditions

1. $(A\circ\varphi^4)^n$ collapses L_{coll} on p or in r,
2. $p, r \in (A\circ\varphi^4)^n\mathcal{I}_{\varphi^4}$,
3. $(A\circ\varphi^4)^{-i}(p) \cap (A\circ\varphi^4)^{-j}(p) \ne \emptyset$, and
4. $(A\circ\varphi^4)^{-i}(r) \cap (A\circ\varphi^4)^{-j}(r) \ne \emptyset$

are all algebraic. In order to conclude that $\Omega_1^n \cap \Omega_2^n \cap \Omega_3^n$ is a dense open subset of $\mathrm{PGL}(3,\mathbb{C})$ we only need to show that this intersection is not empty. Notice that $\mathrm{id} \in \Omega_1^n \cap \Omega_2^n$ follows from our study of φ^4. For even i, j, Property 3 holds for $A = \mathrm{id}$ because p is in the basin of ∞ for $\varphi^4|L_Y$. Similarly, for odd i, j by taking $\varphi^4(p)$, which is in the basin of ∞ for $\varphi^4|L_X$.

By Baire's Theorem the intersection $\Omega^\infty := \bigcap_n \Omega_1^n \cap \Omega_2^n \cap \Omega_3^n$ is generic in $\mathrm{PGL}(3,\mathbb{C})$.

For $A \in \Omega^\infty$, $A\circ\varphi^4$ is algebraically stable, since $A \notin \Omega_1^n$ for any n. Thus $\lambda_1(A\circ\varphi^4) = 16$. Meanwhile, for any $A \in \mathrm{PGL}(3,\mathbb{C})$, $\lambda_2(A\circ\varphi^4) = \lambda_2(\varphi^4) = 16$. Thus, for any $A \in \Omega^\infty$, $A\circ\varphi^4$ is non-cohomologically hyperbolic.

We will now verify that for $A \in \Omega^\infty$ the composition $A\circ\varphi^4$ satisfies the hypotheses of Proposition 4.1. Condition (1) follows with $k = 1$ from Lemma 3.2 since φ^4 blows-up p to C, which is singular at s. Thus, $A\circ\varphi^4$ blows-up p to $A(C)$, which is singular at $A(s)$. Since $A \in \Omega_1^n \cup \Omega_2^n \cup \Omega_3^n$ for any n, the point p has an infinite pre-orbit under $A\circ\varphi^4$ along which $A\circ\varphi^4$ is finite. Thus, Condition (2) holds. Finally, since φ^4 is finite at r with $\varphi^4(r) = s$, the composition $A\circ\varphi^4$ is finite at r and maps r to $A(s)$. Condition (3) then follows from the fact that $A \in \Omega_1^n \cup \Omega_2^n \cup \Omega_3^n$.

Since the hypotheses of Proposition 4.1 are satisfied, no iterate of $A\circ\varphi^4$ preserves a foliation.

Observation: All the arguments above hold with φ substituting φ^4, except for the verification of Condition (3), because it is not clear how to ensure that $(A \circ \varphi)^4(p)$ remains singular and has an infinite pre-orbit along which $A \circ \varphi$ is finite. For the sake of simplicity we decided not to address this technicality, but we expect that for a generic rotation of φ the same result holds.

Question. *(Ch. Favre) Let $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ be an algebraically stable map that is not cohomologically hyperbolic. Then generic rotations $A \circ \eta$ will have the same dynamical degrees and hence also be non-cohomologically hyperbolic. Under what conditions on η will generic rotations of η not preserve a foliation?*

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