My primary research interest is dynamical systems in several complex variables. In general, I like problems and situations that lend themselves well to experimentation, either through specific, simplified examples or computer simulation. I am particularly interested in problems that have a physical interpretation or a real world application. I will briefly describe my completed results in Section 1. Section 2 is devoted to my active projects and plans for future research. Sections 1.4 and 2.4 describe projects with undergraduate students.

1 PREVIOUS WORK

1.1 Basic Concepts of Complex Dynamics in Higher Dimension

There are several good sources for dynamics in one complex variable, for example [24] and [2]. The iteration of holomorphic mappings of the complex numbers, $\mathbb{C}$, has been a very active field in the last thirty years, and a great number of advances have been made.

Generalizing complex dynamics to higher dimensions is quite difficult, with the majority of the progress done for rational maps in two variables. When working with rational functions in one variable, one typically works on the Riemann sphere, so that the poles can be iterated. To iterate rational mappings of two complex variables, instead of iterating functions from $\mathbb{C}^2$ to $\mathbb{C}^2$, it is more common to iterate functions $f\colon \mathbb{CP}^2 \to \mathbb{CP}^2$, where $\mathbb{CP}^2$ is the complex projective plane. This complex projective plane plays an analogous role to that played by the Riemann Sphere $\hat{\mathbb{C}}$ in one dimension.

Similarly, if working in n dimensions, one usually works on the complex projective space $\mathbb{CP}^n$ rather than $\mathbb{C}^n$. Let f be a rational map of $\mathbb{CP}^n$. A point p is in the Fatou set of f if there is a neighborhood U of p so that the iterates $\{f, f^2, \dots\}$ form a normal family on U. The Julia set $J(f)$ is the complement of the Fatou set. These are the same definitions used when dealing with one complex variable, but basic definitions aside, the generalization of this theory to several variables has been complicated by the fact that many of the standard tools and techniques used in the one variable case simply do not work. An example of one such complication that comes up rather quickly is that the degree of a rational mapping of $\mathbb{CP}^2$ may not equal the iterated degree [14]. Another is the failure of Montel's theorem in higher dimension; its most common replacement in terms of the Kobayashi metric is very hard to adapt to many dynamical situations. A more effective replacement for Montel's Theorem is compactness theorems from pluripotential theory. However, they are still less simple to use than Montel's Theorem. A nice survey of the subject includes [13], [14], [26], and [8].

My long term goals are to help develop new techniques to understand the problems involving higher dimension complex dynamics. I have divided most of my time over the past several years between learning modern techniques and working on the projects described in the next two sections.

1.2 Meromorphic Maps and Invariant Foliations

Topological entropy of a map is a quantification of the dynamical complexity of a map. In [15], Gromov established the idea of dynamical degrees as a way to provide an upper bound on entropy. Dynamical degrees measure the growth of volumes of complex subvarieties [8]. Specifically, a meromorphic map $\phi : X \dashrightarrow X$ of a compact Kähler manifold X induces a well-defined pullback action $\phi^* : H^{p,p}(X) \to H^{p,p}(X)$ for each $1 \le p \le \dim(X)$. The p-th dynamical degree is

$$\lambda_p(\phi) := \lim_{n\to\infty} \|(\phi^n)^* : H^{p,p}(X) \to H^{p,p}(X)\|^{1/n}$$

One says that ϕ is *cohomologically hyperbolic* if one of the dynamical degrees is strictly larger than all of the others. In this case, there is a conjecture [16, 18] about the ergodic properties of ϕ. This conjecture has been proved in several particular sub-cases [17, 7, 10].

It was shown in [6] that bimeromorphic maps of surfaces that are not cohomologically hyperbolic ($\lambda_1(\phi) = \lambda_2(\phi) = 1$) always preserve an invariant fibration. If a meromorphic map preserves a fibration, then there are nice formulae relating the dynamical degrees of the map [9, 11]. Based on this evidence, Guedj asked in [18, p. 103] whether every non-cohomologically hyperbolic map preserves a fibration.

We proved:

Theorem 1.1 (K, Roeder, Perez)**.** *The rational map* $\phi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ *given by*

$$\phi[x : y : z] \ = \ [-y^2 : x(x-z) : -(x+z)(x-z)] \tag{1.1}$$

is not cohomologically hyperbolic, and no iterate of ϕ *preserves a singular holomorphic foliation.*

Since preservation of a fibration is a stronger condition than preserving a singular foliation, ϕ provides an answer to the question posed by Guedj. This result is described in [21].

After reading a preliminary version of this paper, Charles Favre asked is the same behavior can be found for a polynomial map of $\mathbb{C}^2$. In [12, §7.2], there is a list of seven types of non-cohomologically hyperbolic polynomial mappings. Our method of proof does not apply in most of these examples for trivial reasons, but we found a map of type (3) for which the same result holds:

Theorem 1.2 (K, Roeder, Perez)**.** *The polynomial map*

$$\psi(x, y) := \big(x(x-y) + 2, (x+y)(x-y) + 1\big)$$

extends as a rational map $\psi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ *that is not cohomologically hyperbolic (*$\lambda_1(\psi) = \lambda_2(\psi) = 2$*), and no iterate of* ψ *preserves a singular holomorphic foliation on* $\mathbb{CP}^2$*.*

1.3 Symmetrization in $\mathbb{CP}^k$

Symmetric products have been used to produce simple examples of endomorphisms of $\mathbb{CP}^k$, $k \geq 2$ as dynamical systems [8]. We use those a priori simple maps to generate interesting, nontrivial examples in higher dimension. This project was begun at an AIM workshop on post-critically finite (PCF) maps. These PCF maps have become a very handy tool in understanding dynamics in higher dimension. One of the goals at the workshop was to find maps that are PCF "all the way down." More specifically, we say that $F \in \text{End}_d(\mathbb{CP}^k)$ is 1-deep postcritically finite if F is postcritically finite and $F_{(1)} := F|_{\mathcal{C}_F}$ is postcritically finite, meaning the orbit under iteration of F of the critical locus $\mathcal{C}(F_{(1)})$ of $F_{(1)}$ is an algebraic subvariety of pure codimension 2. We say that F is $(j+1)$-deep postcritically finite if it is j-deep postcritically finite and $F_{(j+1)} := F|_{\mathcal{C}(F_{(j)})}$ is postcritically finite. We say that F is strongly postcritically finite if it is $(k-1)$-deep postcritically finite.

Specifically, we have the following theorems:

Theorem 1.3 (K, Gauthier)**.** *If* F *is the k-symmetric product of* f*, a map* $\mathbb{CP}^1$*, then* F *is strongly post-critically finite if and only if* f *is post-critically finite.*

Theorem 1.4 (K, Gauthier)**.** *Let* f *be a Lattés map of* $\mathbb{CP}^1$*. Then its k-symmetric product* F *is a Lattés map of* $\mathbb{CP}^k$*.*

We expect this symmetrization process to also produce isolated Lattés maps of $\mathbb{CP}^k$, which could be used to enlighten bifurcation phenomena in parameter spaces. The paper describing these results is currently in preparation.

1.4 Geometric Limits of Julia Sets

In this single complex variable project, I had the pleasure of working with two very talented undergraduate students, Reaper Romero and David Simmons, and the main result can be found in [23]. Consider the family of maps

$$P_{n,c}(z) \ = \ z^n + c,$$

where $n \geq 2$ is an integer and $c \in \mathbb{C}$ is a parameter. The filled Julia set $K(P_{n,c})$ is the set of points that stay bounded under iteration by $P_{n,c}$, and its boundary is $J(P_{n,c})$, the Julia set. In [5], the structure of these sets as $n \to \infty$ was examined. One of the major results is this work was

Theorem 1.5 (Boyd-Schwarz, 2011)**.** *Let $c \in \mathbb{C}$.*

(1) If $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$, then

$$\lim_{n\to\infty} J(P_{n,c}) = \lim_{n\to\infty} K(P_{n,c}) = S^1.$$

(2) If $c \in \mathbb{D}$, then

$$\lim_{n\to\infty} J(P_{n,c}) = S^1 \text{ and } \lim_{n\to\infty} K(P_{n,c}) = \overline{\mathbb{D}}.$$

(3) If $c \in S^1$, then if $\lim_{n\to\infty} J(P_{n,c})$ and/or $\lim_{n\to\infty} K(P_{n,c})$ (and/or any liminf or limsup) exists, it is contained in $\overline{\mathbb{D}}$.

The purpose of this project was to improve part (3), and the results of the project can be found in [23]. While there may be no Hausdorff limit as $n \to \infty$ for $J(P_{n,c})$ or $K(P_{n,c})$, experimentation suggested that given $c \in S^1$, there is a predictable pattern for $K(P_{n,c})$ as n increases. $K(P_{n,c})$ is connected and full or totally disconnected, depending on whether the orbit of the critical point 0 stays bounded or not, respectively. For the family of maps where $c \in S^1$, we have $P_{n,c}(0) = c \in S^1$. It turns out, though, that the pattern of $K(P_{n,c})$ as n increases is determined by the behavior of critical orbit as it leaves the circle. More specifically, we proved the following result:

Theorem 1.6 (K, Romero, Simmons)**.** *Let $c = e^{2\pi i\theta} \in S^1$ such that $\theta \neq 0$ and $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} J(P_{n,c}) \text{ and } \lim_{n\to\infty} K(P_{n,c})$$

do not exist. Moreover, if θ is rational, $\theta \neq 0$, and $\theta \neq \frac{3q\pm1}{3(6p-1)}$, then there exist N and subsequences a_k and b_k partitioning $\{n \in \mathbb{N} \colon n \geq N\}$ such that

$$\lim_{k\to\infty} K(P_{a_k,c}) \;=\; S^1 \qquad \text{and} \qquad \lim_{k\to\infty} K(P_{b_k,c}) \;=\; \overline{\mathbb{D}}.$$

1.5 Superstable Manifolds for Invariant Circles

The main results in [22], a joint work with my Ph.D. advisor, Roland Roeder, were a culmination of nearly two years of work. The proof involved techniques from several fields including rudimentary algebraic geometry, several complex variables, and real dynamical systems.

Let f be a dominant meromorphic self-map of a compact connected Hermitian manifold X of dimension $n > 1$. For simplicity, one can think of a rational map $f \colon \mathbb{CP}^n \to \mathbb{CP}^n$ with $n > 1$. The focus is on the situation where there is $L \subset X$, an embedded copy of $\mathbb{CP}^1$, with f holomorphic in a neighborhood of L. Moreover, L is invariant and transversely superattracting of degree a, and $f|L$ conjugate to $z \mapsto z^b$. Although this is a rather special situation, it has appeared in examples from [25, 19, 4, 3]. For such maps, the Julia set of $f|L$ is an invariant circle $S = \{|z| = 1\}$, which is a hyperbolic set for f. Let N be an open neighborhood of L such $f \colon N \to N$ is holomorphic. Given a neighborhood Ω of S, the local stable manifold

$$\mathcal{W}^s_{\text{loc}}(S) \;=\; \{x \in N \colon f^n x \in \Omega \text{ and } f^n x \to S \text{ as } n \to \infty\} \tag{1.2}$$

is a real $2n-1$ dimensional manifold.

This set is relevant dynamically as it separates the two basins of attraction for the fixed points in L. Thus, $\mathcal{W}^s_{\text{loc}}(S)$ is contained in the Julia set for f, and very little is currently known about the structure and behavior of Julia sets in dimension greater than one. We proved

Theorem 1.7 (K, Roeder)**.** *If $a \geq b$, then $\mathcal{W}^s_{\text{loc}}(S)$ has real analytic regularity.*

We also show that the condition that $a \geq b$ cannot be improved without adding additional hypotheses. Consider the mapping $\mathcal{R} \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\mathcal{R}(z,t) \;=\; \left(\frac{z^2(z^2+t^2)}{1+t^2z^2}, \frac{t^2(z^2+1)^2}{(z^2+t^2)(1+t^2z^2)}\right). \tag{1.3}$$

The complex line $\mathcal{L}_0 := \{t = 0\}$ is invariant under $\mathcal{R}$ and transversely superattracting with degree 2. Within this line, the map $\mathcal{R}$ is given by $z \mapsto z^4$. Therefore, the circle

$$\mathcal{B} = \{|z| = 1, t = 0\} \subset \mathcal{L}_0, \tag{1.4}$$

is invariant under $\mathcal{R}$, attracting in the t direction, and expanding in the z direction. This situation satisfies the hypotheses for Theorem 1.7 with the exception that $2 = a < b = 4$. The stable manifold, $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$, was previously shown to be C^∞ by Bleher, Lyubich, and Roeder [4, 3]. However, in [22], we proved

Theorem 1.8 (K, Roeder)**.** *The stable manifold $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ is not real analytic at any point.*

The skew product map $f\colon \mathbb{C}^2 \to \mathbb{C}^2$, given by

$$f(z, w) = (z^3 + 2wz^2, w^2) \tag{1.5}$$

similarly satisfies the hypotheses of Theorem 1.7 with the exception that $2 = a < b = 3$ and produces a stable manifold $\mathcal{W}^s_{\text{loc}}(S)$ that is not real analytic. This example and Theorem 1.8 show that the result in Theorem 1.7 is sharp.

1.6 Super-stable Manifolds for the Renormalization on the DHL

Theorem 1.8 may seem like an oddly specific example, except that it is connected with statistical physics. Before describing the proof of Theorem 1.8, let me explain this physical motivation. The Ising model is a mathematical model of magnetism that relates magnetic properties of some given matter to a sequence of graphs Γ_n. Associated to each of these graphs is a partition function $\mathsf{Z}_n(z, t)$, where $z = e^{-h/T}$ and $t = e^{-2J/T}$. Here $J > 0$ is the energy of interaction, T is temperature, and h is some externally applied magnetic field. The partition function $\mathsf{Z}_n(z, t)$ has zeros

$$\mathcal{S}^c_n := \{(z, t) \in \mathbb{C}^2 \,:\, \mathsf{Z}_n(z, t) = 0\}$$

that describe the singularities of the Ising model associated to Γ_n. These are called the *Lee-Yang-Fisher zeros*, and the limiting distribution of Lee-Yang zeros as $n \to \infty$ is believed to describe the real microscopic physics. For background on the Ising model, see [1].

Now consider the Diamond Hierarchal Lattice (DHL), whose construction is illustrated in Figure 1. This Ising model on the DHL was studied in great detail in [4] and [3].

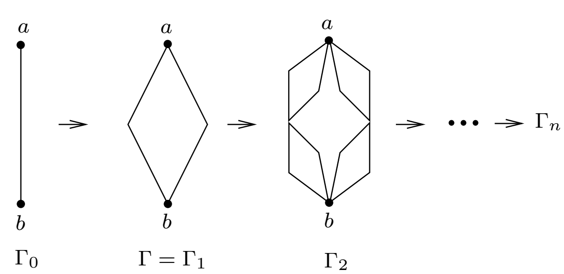

Figure 1: Diamond hierarchical lattice (DHL).

It is proved in [4] that the limiting distribution of the Lee-Yang-Fisher zeros for the DHL exists as a closed, positive $(1,1)$-current $\mathcal{S}^c$ on $\mathbb{P}^2$. In fact, $\mathcal{S}^c = \frac{1}{2}\Psi^* S$, where S is the Green current for the map $R\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ given in homogeneous coordinates by

$$R[U : V : W] = [(U^2 + V^2)^2 : V^2(U + W)^2 : (W^2 + V^2)^2].$$

R is semi-conjugate to $\mathcal{R}$ by means of a rational map $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$. The advantage of using R is that it resolves two indeterminant points and thus, has better global dynamics. The support of $\mathcal{S}^c$ describes locus where phase transitions occur in $\mathbb{C}^2$.

It is shown in [4] that at low complex temperatures supp $\mathcal{S}^c$ coincides with $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$. Combining Theorem 1.8 with the work from [4] gives the following:

Corollary 1.9 (K, Roeder)**.** *At low complex temperatures ($|t|$ small), the locus of phase transitions for the Ising model on the DHL is a* 3 *real-dimensional manifold that is C^∞ but not real analytic.*

To our knowledge, this is the first result of this kind.

2 CURRENT PROJECTS

2.1 Meromorphic Maps that Preserve No Foliation

The general strategy for proving Theorems 1.1 and 1.2 seems to be fairly robust. It is expected that many more examples can be generated in the same way. However, ruling out the resonance of dynamical degrees, there is no clear explanation for why some maps should preserve a foliation and others should not.

Moreover, whether or not maps that are not cohomologically hyperbolic and preserve no foliation are plentiful, little is known about their ergodic properties. In particular,

Question 1. Does the map ϕ have a unique measure of maximal entropy?

We are able to show that Theorem 1.2 still holds for ψ_a, a parameterization of the polynomial mapping ψ. Moreover, for a of large enough modulus, we know that $J(\psi_a)$ is a dyadic Cantor set, so ϕ_a has entropy $\log 2$. The goal of the project moving forward is to construct another measure of maximal entropy using a different method.

2.2 Bifurcation Shadowing

This project was motivated by a question raised by Laura DeMarco at the 2013 AMS MRC on Complex Dynamics. The family of quadratic rational maps of $\mathbb{C}$ that have a fixed point with multiplier λ and critical points ± 1can be parameterized by

$$f_a(z) = \frac{\lambda z}{z^2 + az + 1}$$

Their are particular values of λ where the bifurcation loci associated to each of the two critical points nearly coincide.

Question 2. What is the source of the similarity between the two bifurcation loci?

In joint work with Dan Cuzzucreo and Joanna Furno, we suspect that those values of λ correspond to maps in which one of the critical points is strictly preperiodic. Progress in this project has come from considering this as a two-parameter family and examining slices of the bifurcation loci in $\mathbb{C}^2$.

2.3 Regularity of Superstable Manifolds When $b > a$

Theorem 1.7 naturally leads one to question whether there are necessary conditions for $\mathcal{W}^s_{\text{loc}}(S)$ to be real analytic. If $a < b$, the superattracting direction is not as strong as the expansion within L; I believe there should be a way to exploit this to answer the following question:

Question 3. Is there a "generic" class of mappings f with $a < b$ for which $\mathcal{W}^s_{\text{loc}}(S)$ is not real analytic?

Here I am trying to generalize of the technique in Section 1.1.6 for the Migdal-Kadanoff renormalization that proves $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ is not real analytic. If f happens to be a product, the stable manifold will be real analytic, so the class of functions not producing a real analytic stable manifold is at best generic in some sense. The technique used for $\mathcal{R}$ (given by(1.3)) and f (given by (1.5)) relies on a second invariant line L_1 such that $f \mid L_1$ is the map $w \mapsto w^a$. I suspect one may use the degree a transversal superattraction of L to generate an invariant cone field to serve the same purpose in the general case.

Question 4. For any a and b, is $\mathcal{W}^s_{\text{loc}}(S)$ a C^∞ manifold?

Following the method in [4, Proposition 9.12], define the sequence $B_n(x) := \frac{1}{b^n} Df^n(x)$. It is not difficult to show B_n converges uniformly on compact subsets of $\mathcal{W}^s_{\text{loc}}(S)$ at super-exponential rate to a matrix-valued function $B(x)$. The goal is to prove this function B is C^∞ in any neighborhood of S, since one can use the invariance

$$bB_n(x) = B_{n-1}(f(x))Df(x) \text{ and } bB(x) = B(f(x))Df(x), \tag{2.1}$$

to extend the result to any compact subset of $\mathcal{W}^s(S)$. $B(x)$ and all of its derivatives are converging so fast that I believe Whitney's extension theorem could be used to extend it to a C^∞ function in a neighborhood of S. In this case, since L is transversely superattracting, $\ker B(x)$ would be a C^∞ holomorphic line field, so that one could integrate it to get a C^∞ foliation by holomorphic discs of S. Within $\mathcal{W}^s_{\text{loc}}(S)$, $\ker(B)$ is the field of tangent planes to the Levi foliation of $\mathcal{W}^s_{\text{loc}}(S)$. Therefore, $\mathcal{W}^s_{\text{loc}}(S)$ is formed as a union of the holomorphic discs from the C^∞ foliation.

2.4 New Hierarchical Lattices

I presented Theorem 1.7 at the 2014 Western Spring Sectional AMS Meeting, and Jacopo di Simoi asked a simple, but interesting question.

Question 5. Are there any other lattices besides the DHL for which Theorem 1.7 is true?

See Figure 1 in Section 1.6, where the Ising Model is described. This quickly becomes a combinatorial problem with hypotheses to check in $\mathbb{CP}^n$. Also, the DHL is built on using only two states; one could ask

Question 6. Under what conditions does Theorem 1.7 hold with more than two states?

This project is moving forward this fall with Matthew Campbell, an undergraduate mathematics major at University of Arizona.

2.5 Lee-Yang Density

The Lee-Yang zeros are obtained from the Lee-Yang-Fisher zeros by requiring that $t \in [0, 1]$, which correspond to "physical" temperatures. The Lee-Yang Circle Theorem asserts that for each n and fixed $t_0 \in [0, 1]$, zeros of partition function $\mathsf{Z}_n(z, t_0)$ corresponding to Γ_n lie on the unit circle $\mathbb{T}_{t_0} := \{|z| = 1, t = t_0\}$.

The advantage of studying the DHL with $\mathcal{R}\colon \mathbb{C}^2 \to \mathbb{C}^2$, which is given by (1.3), is that $\mathcal{R}$ gives an explicit recursion for the Lee-Yang zeros. In particular, $\mathcal{R}$ maps the Lee-Yang cylinder $\mathcal{C} := \{|z| = 1, t \in [0, 1]\}$ into itself, with the Lee-Yang zeros corresponding to Γ_{n+1} obtained by pulling back the Lee-Yang zeros corresponding to Γ_n under $\mathcal{R}|\mathcal{C}$. This recursion was examined thoroughly to give a characterization of the limiting distribution of the Lee-Yang zeros for the DHL in the recent preprints [4, 3] by Bleher, Lyubich, and Roeder.

At temperatures $t < t_c \approx 0.2956$, the description in [4] is given as follows. Through each point $x \in \mathcal{B}$, is a stable manifold, $\mathcal{W}^s_{\text{loc}}(x)$, and their union forms a foliation, $\mathcal{F}^s$, in a neighborhood of $\mathcal{B}$ within $\mathcal{C}$. By construction, $\mathcal{F}^s$ is obtained by intersecting the Levi foliation of $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ with the cylinder $\mathcal{C}$. Note that these stable manifolds are obtained as the intersection of the Lee-Yang cylinder $\mathcal{C}$ and the local stable manifolds $\mathcal{W}^s_{\text{loc}}(x)$, which were previously considered in $\mathbb{C}^2$, where they are holomorphic discs. The limiting distribution of Lee-Yang zeros at temperature $0 < t_0 < t_c$ is obtained by the pushforward of Lebesgue measure from $\mathcal{B}$ to the circle $\{t = t_0\}$ in $\mathcal{C}$ along the foliation $\mathcal{F}^s$. It is shown in [3, Lemma 3.2] that $\mathcal{F}^s$ has the same regularity that the stable manifold $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ does as a submanifold of $\mathbb{C}^2$. Therefore, Theorem 1.8 implies

Corollary 2.1. *For any $z = e^{i\phi} \in \mathcal{B}$, there is a dense set of $t_0 \in [0, t_c]$ so that the limiting distribution of Lee-Yang zeros within $\mathbb{T}_{t_0}$ does not have real analytic density at (t_0, ϕ).*

Unfortunately, the fact that $\mathcal{F}^s$ not real analytic at any point does not imply that none of the non-trivial holomomies are real analytic. Isakov [20] proved a similar result for Ising models on the $\mathbb{Z}^d$ lattice. However, Isakov's result required a great deal of difficult and complicated analysis. I would like to prove the analogus result:

Conjecture 1. *For temperature $0 < t < t_c$, the limiting density of Lee-Yang zeros for the DHL $\rho_t(\phi)$ is C^∞, but not real analytic.*

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