

# SUPERSTABLE MANIFOLDS OF INVARIANT CIRCLES AND CO-DIMENSION 1 BÖTTCHER FUNCTIONS

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ABSTRACT. We consider the situation of a dominant meromorphic self-map  $f : X \dashrightarrow X$ , where  $X$  is a compact Kähler manifold of dimension  $n > 1$ . Suppose there is an embedded copy of  $\mathbb{P}^1$  that is invariant under  $f$ , with  $f$  holomorphic and transversally superattracting with degree  $a$  in some neighborhood. Suppose  $f$  restricted to this line is given by  $z \mapsto z^b$ , with resulting invariant circle  $S$ . We prove that if  $a \geq b$ , then the local stable manifold  $\mathcal{W}_{\text{loc}}^s(S)$  is real analytic. In fact, we state and prove a suitable localized version that can be useful in wider contexts. We then show that the condition  $a \geq b$  cannot be relaxed without adding additional hypotheses by presenting two examples with  $a < b$  for which  $\mathcal{W}_{\text{loc}}^s(S)$  is not real analytic in the neighborhood of any point.

## 1. INTRODUCTION

Let  $f : X \dashrightarrow X$  be a dominant meromorphic self-map of a compact connected Kähler manifold  $X$  of dimension  $n > 1$ . Here, the focus is on the situation in which there is  $L \subset X$ , an embedded copy of  $\mathbb{P}^1$ , with  $f$  holomorphic in a neighborhood of  $L$ ,  $L$  is invariant and transversally superattracting of degree  $a$ , and  $f|L$  conjugate to  $z \mapsto z^b$ . Although this is a rather special situation, it has appeared in examples from [24, 11, 5, 4]. For such maps, the Julia set of  $f|L$  is an invariant circle  $S$ , which is a hyperbolic set for  $f$ . The local stable manifold  $\mathcal{W}_{\text{loc}}^s(S)$  is a real  $2n - 1$  dimensional manifold. We will prove:

**Theorem 1.** *If  $a \geq b$ , then  $\mathcal{W}_{\text{loc}}^s(S)$  has real analytic regularity.*

To prove the theorem, we will localize to the situation to a tubular neighborhood of  $L$  which is forward invariant under  $f$ . This tubular neighborhood will be biholomorphic to a neighborhood of the zero section of the normal bundle to  $L$ . Theorem 1 is a direct consequence of the following:

**Theorem 1'.** *Consider any holomorphic  $\mathbb{C}^{n-1}$  bundle over  $\mathbb{P}^1$  with zero section  $L$ . Let  $N$  be an open neighborhood of  $L$  and  $f : N \rightarrow N$  a holomorphic map. Suppose  $L$  is invariant and transversally superattracting with degree  $a$  and that  $f|L$  is conjugate to  $z \mapsto z^b$  with invariant circle  $S$ . If  $a \geq b$ , then  $\mathcal{W}_{\text{loc}}^s(S)$  has real analytic regularity.*

For diffeomorphisms, the existence and regularity of the local stable manifold for a hyperbolic invariant manifold  $V$  was studied extensively by Hirsch, Pugh, and Shub [10]. In order to guarantee existence of a  $C^r$  local stable manifold, they assume a strong form of hyperbolicity known as *r-normal hyperbolicity*. Specifically, they require that the rate of expansion in the unstable direction transverse to  $V$  dominates the  $r$ -th power of the maximal rate of expansion tangent to  $V$  and that the rate of contraction in the stable direction transverse to  $V$  dominates the  $r$ -th power of the maximal rate of contraction tangent to  $V$ .

Although the maps considered in this paper are many-to-one, they also do not fit in the context of [10] since  $f|L$  is conformal, forcing that the rates of expansion tangent to  $S$  and transverse to  $S$  are equal. In particular, [10] only guarantees that  $\mathcal{W}_{\text{loc}}^s(S)$  is of  $C^0$  regularity.

In §2 we prove Theorem 1' by constructing a semi-conjugacy between  $f$  and  $z \mapsto z^b$  on a forward invariant neighborhood of  $S$ . The construction is rather analogous to the proof of Böttcher's Theorem from one-dimensional complex dynamics [6]; see also [18, Ch. 9]. Since our construction is in higher dimensions, we call it a “co-dimension 1” Böttcher function. The related problem of

local conjugation of a holomorphic map to its lowest order terms near a superattracting fixed point has been considered in higher dimensions by [7].

This is followed by §3, where it is shown that Theorem 1 applies to the maps considered in [24] and the example from §6.2 of [11]. We also present one additional example to which it applies.

In §4, we show that the condition that  $a \geq b$  cannot be improved without adding additional hypotheses. We'll consider two maps for which  $a < b$  and  $\mathcal{W}_{\text{loc}}^s(S)$  is not analytic. One of them is the Migdal-Kadanoff renormalization map  $R$  for the Ising model on the Diamond Hierarchical Lattice (DHL) that was studied extensively in [5, 4]. It has  $a = 2$  and  $b = 4$ . The other is a polynomial skew product with  $a = 2$  and  $b = 3$ .

Let us comment a bit more about the map  $R$ . For this map, the invariant circle  $S$  has the physical context of being related to the bottom of the Lee-Yang cylinder, so it is denoted  $B$ . In [4, Lemma 3.2], the authors proved that  $\mathcal{W}_{\text{loc}}^s(B)$  is a  $C^\infty$  manifold. We prove:

**Theorem 2.** *The stable manifold  $\mathcal{W}_{\text{loc}}^s(B)$  is not real analytic at any point.*

This is of physical interest, since  $\mathcal{W}_{\text{loc}}^s(B)$  is related to phase transitions of the Ising model on the DHL at low temperatures; see [5, 4]. In §5, we'll explain how Theorem 2 relates to the limiting distribution of Lee-Yang and Lee-Yang-Fisher zeros at low temperatures.

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## 2. PROOF OF THEOREM 1'

The  $\mathbb{C}^{n-1}$  bundle over  $\mathbb{P}^1$  can be described by two systems of locally trivializing coordinates  $(z, \mathbf{w}) \in \mathbb{C} \times \mathbb{C}^{n-1}$  and  $(\zeta, \boldsymbol{\omega}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ . For  $z \neq 0$ , they are related by  $\zeta = 1/z$  and  $\boldsymbol{\omega} = A_z \mathbf{w}$ , with  $A_z : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$  a linear isomorphism depending holomorphically on  $z$ . Let us choose these trivializations so that the dynamics on the zero section is  $z \mapsto z^b$ .

We will make use of standard multi-index notation. Given  $\mathbf{c} \in \mathbb{Z}_+^{n-1}$  and  $\mathbf{w} \in \mathbb{C}^{n-1}$ ,  $\mathbf{w}^{\mathbf{c}} = w_1^{c_1} w_2^{c_2} \cdots w_{n-1}^{c_{n-1}}$  and  $|\mathbf{c}| = c_1 + \cdots + c_{n-1}$ . We will always use the standard Hermitian norm  $|\mathbf{w}| = (|w_1|^2 + \cdots + |w_{n-1}|^2)^{1/2}$  on  $\mathbb{C}^{n-1}$ .

**Lemma 3.** *There are holomorphic functions  $\mathbf{g}_1$  and  $\mathbf{g}_{\mathbf{c}}$  for each  $|\mathbf{c}| = a$  such that in the  $(z, \mathbf{w})$  coordinates*

$$f(z, \mathbf{w}) = \left( z^b + \mathbf{w} \cdot \mathbf{g}_1(z, \mathbf{w}), \sum_{|\mathbf{c}|=a} \mathbf{w}^{\mathbf{c}} \mathbf{g}_{\mathbf{c}}(z, \mathbf{w}) \right).$$

*Similarly, there are holomorphic functions  $\mathbf{h}_1$  and  $\mathbf{h}_{\mathbf{c}}$  for each  $|\mathbf{c}| = a$  such that in the  $(\zeta, \boldsymbol{\omega})$  coordinates*

$$f(\zeta, \boldsymbol{\omega}) = \left( \zeta^b + \boldsymbol{\omega} \cdot \mathbf{h}_1(\zeta, \boldsymbol{\omega}), \sum_{|\mathbf{c}|=a} \boldsymbol{\omega}^{\mathbf{c}} \mathbf{h}_{\mathbf{c}}(\zeta, \boldsymbol{\omega}) \right).$$

*Proof.* The proof is the same in both coordinate systems, so we'll work in the  $(z, \mathbf{w})$  system. Since  $f|_L$  is the map  $z \mapsto z^b$ , the first coordinate of  $f$  minus  $z^b$  vanishes on  $L$ . Since  $L$  is given by

$\mathbf{w} = \mathbf{0}$ , we have that the first coordinate of  $f$  is  $z^b + \mathbf{w} \cdot \mathbf{g}_1(z, \mathbf{w})$  for some holomorphic function  $\mathbf{g}_1$ . Meanwhile, the expression for the second coordinate follows from the fact that  $L$  is transversally superattracting of degree  $a$ .  $\square$

**2.1. Hyperbolic theory.** We'll now verify that the local stable manifold  $\mathcal{W}_{\text{loc}}^s(S)$  is a  $2n - 1$  real-dimensional topological manifold that is foliated by local stable manifolds of each point of  $S$ .

The hyperbolic theory for endomorphisms is somewhat less standard than for diffeomorphisms. Suitable references from the context of complex dynamics include [2, 9, 13]. For consistency, we will use definitions and results from [2, Appendix B]. Let us consider the natural extension

$$\hat{S} := \{(x_i)_{i \leq 0} : x_i \in S \text{ and } f(x_i) = x_{i+1}\}.$$

We'll denote such histories by  $\hat{x} = (x_i)_{i \leq 0} \in \hat{S}$ . Notice that the action of  $f$  naturally lifts to an action  $\hat{f} : \hat{S} \rightarrow \hat{S}$ .

**Lemma 4.**  *$S$  is a hyperbolic set for the map  $f$ .*

*Proof.* Note that for  $x \in S$ , we have

$$Df_x = \begin{bmatrix} bz^{b-1} & \frac{\partial}{\partial \mathbf{w}} \mathbf{g}_1(z, \mathbf{0}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus, we have  $E^s(x) = \ker(Df)$  and  $E^u(\hat{x}) \subset L$ , so  $T_x \mathbb{C}^n = E^s(x) \oplus E^u(\hat{x})$ . Invariance of  $E^s(x)$  follows from the fact any point in the kernel is collapsed to  $(0, \mathbf{0})$  under  $Df$ , and invariance of  $E^u(\hat{x})$  follows from the invariance of  $L$ . Also, for any  $v^s \in E^s(x)$  and  $v^u \in E^u(\hat{x})$  with  $n \geq 0$ ,

$$\|Df_x^n v^s\| = 0 \leq C\lambda^n \|v^s\| \text{ and } \|Df_x^n v^u\| \leq C\lambda^{-n} \|v^u\|,$$

for  $C = 1$  and  $\lambda = 1/2$ . Thus, we have that  $S$  is hyperbolic.  $\square$

Therefore, by the stable manifold theorem (see, for example, [21, Theorem 5.2]) each point  $x \in S$  will have local stable manifold  $\mathcal{W}_{\text{loc}}^s(x)$  that is a complex  $n - 1$  ball holomorphically embedded into  $N$  and each prehistory  $\hat{x}$  will have a local unstable manifold  $\mathcal{W}_{\text{loc}}^u(\hat{x})$ , which is a holomorphic disc. They depend continuously on  $x$  and  $\hat{x}$ . (In this case, the unstable manifolds all lie in  $L$ .)

*Remark.* Existence of the stable manifolds for each point of  $S$  can also be proved in the following simpler way, which is a direct generalization of what was done in [24, Proposition 4.2] and [5, Proposition 9.2].

By the stable direct manifold theorem for a point (see, for example, [20, §2.6], which holds even if  $Df$  has an eigenvalue of 0), there exists a local stable manifold for  $\mathcal{W}_{\text{loc}}^s((1, \mathbf{0}))$ , which is the graph of a holomorphic function  $z = \eta(\mathbf{w})$  defined on some open ball in the  $\mathbf{w}$  axis. Using a suitable invariant cone field and a well-chosen neighborhood of  $S$ , one can take iterated preimages of this stable manifold, each of which is expressed as the graph of a holomorphic function defined on the same ball. In this way, we construct local stable manifolds over the set of iterated preimages of  $(1, \mathbf{0})$ , which is dense in  $S$ . Finally, one can apply the holomorphic lambda lemma [17, 16] to extend to the closure, obtaining stable manifolds for every point of  $S$ .

**Definition 1.** A hyperbolic set  $\hat{\Lambda}$  has a local product structure, if  $\delta > 0$  can be chosen small enough so that for any  $p \in \Lambda$  and  $\hat{q} \in \hat{\Lambda}$ , either  $\mathcal{W}_\delta^s(p) \cap \mathcal{W}_\delta^u(\hat{q})$  is empty or it is a single point  $x \in \Lambda$  so the unique history  $\hat{x}$  of  $x$  satisfying  $x_j \in \mathcal{W}_\delta^u(\hat{f}^j(\hat{q}))$  for all  $j \leq 0$  is completely contained in  $\hat{\Lambda}$ .

**Lemma 5.**  *$S$  has local product structure for the map  $f$ .*

*Proof.* By Lemma 4,  $S$  is hyperbolic. Recall that for any  $\hat{q} \in \hat{S}$ , we have that  $\mathcal{W}_\delta^u(\hat{q}) = \mathbb{D}_\delta(q_0) \subset L$ , which is the disc of radius  $\delta > 0$  centered at the point  $q$  contained in  $L$ . Since  $\mathcal{W}_\delta^u(\hat{q})$  depends only on  $q_0$ , existence of a local product structure for  $\hat{S}$  is very simple.

By the Stable Manifold Theorem, we may choose  $\delta > 0$  small enough so that for any  $p \in S$ , we have  $\mathcal{W}_\delta^s(p) \cap L = \{p\}$ . Thus, for any two points  $p, q \in S$ , the intersection  $\mathcal{W}_\delta^s(p) \cap \mathcal{W}_\delta^u(\hat{q}) = \{p\}$ , with  $p \in S$ . Moreover,  $p$  has a unique prehistory  $\hat{p} = (p_i)_{i \leq 0}$  with  $p_j \in \mathcal{W}_\delta^u(\hat{f}^j(\hat{q}))$  for all  $j \leq 0$ , and it is completely contained in  $\hat{S}$  as well.  $\square$

Given a neighborhood  $\Omega$  of  $S$ , let

$$(1) \quad \mathcal{W}_{\text{loc}}^s(S) := \{x \in N : f^n x \in \Omega \text{ and } f^n x \rightarrow S \text{ as } n \rightarrow \infty\}$$

(where  $\Omega$  is implicit in the notation, and an assertion involving  $\mathcal{W}_{\text{loc}}^s(S)$  means that it holds for any sufficiently small neighborhood of  $S$ ).

Since  $S$  has a local product structure  $\mathcal{W}_{\text{loc}}^s(S)$  is the union of the local stable manifolds  $\mathcal{W}_{\text{loc}}^s(x)$  of points  $x \in \mathcal{B}$ ; see [2, Proposition B.6]. The local stable manifolds of points are pairwise disjoint and depend continuously on the base point, therefore we have:

**Corollary 6.**  $\mathcal{W}_{\text{loc}}^s(S)$  is a topological manifold of real dimension  $2n - 1$ .

**2.2. Böttcher function.** Let  $(z_n, \mathbf{w}_n) := f^n(z, \mathbf{w})$ . Motivated by Böttcher's theorem [6],[18, p. 86], we consider a sequence of functions

$$\varphi_n(z, \mathbf{w}) = z_n^{1/b^n}.$$

We will show that the  $\varphi_n$  converge uniformly on compact subsets of some forward invariant neighborhood  $\Omega$  of  $S$  to a holomorphic function  $\varphi$  that semi-conjugates  $f$  to  $z \mapsto z^b$ :

$$(2) \quad \varphi(f(z, \mathbf{w})) = \varphi(z, \mathbf{w})^b.$$

To make sense of the  $b^n$ -th roots and the limit, we'll rewrite each  $\varphi_n$  as telescoping product:

$$(3) \quad \varphi = \lim_{n \rightarrow \infty} \varphi_n = z_0 \cdot \frac{z_1^{1/b}}{z_0} \cdot \frac{z_2^{1/b^2}}{z_1^{1/b}} \cdot \frac{z_3^{1/b^3}}{z_2^{1/b^2}} \cdots = z_0 \prod_{n=0}^{\infty} \left( \frac{z_{n+1}}{z_n^b} \right)^{\frac{1}{b^{n+1}}},$$

where it follows from Lemma 3 that

$$(4) \quad \frac{z_{n+1}}{z_n^b} = \frac{z_n^b + \mathbf{w}_n \cdot \mathbf{g}_1(z_n, \mathbf{w}_n)}{z_n^b} = 1 + \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n).$$

In the  $(\zeta, \boldsymbol{\omega})$  coordinates we have:

$$(5) \quad \frac{z_{n+1}}{z_n^b} = \frac{\zeta_n^b}{\zeta_{n+1}} = \frac{1}{1 + \frac{\boldsymbol{\omega}_n}{\zeta_n^b} \cdot \mathbf{h}_1(\zeta_n, \boldsymbol{\omega}_n)}.$$

When working in  $\mathcal{W}^s(\eta_1)$  we'll use expression (4), when working in  $\mathcal{W}^s(\eta_2)$  we'll use expression (5), and when working on  $\mathcal{W}_{\text{loc}}^s(S)$ , we'll use either.

We'll construct a forward invariant neighborhood  $\Omega$  of  $S$  so that if  $(z, \mathbf{w}) \in \Omega \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$ , then

$$(6) \quad \left| \frac{\mathbf{w}_n}{z_n^b} \cdot \mathbf{g}_1(z_n, \mathbf{w}_n) \right| < \frac{1}{2},$$

and if  $(\zeta, \boldsymbol{\omega}) \in \Omega \cap (\mathcal{W}^s(\eta_2) \cup \mathcal{W}_{\text{loc}}^s(S))$ , then

$$(7) \quad \left| \frac{\boldsymbol{\omega}_n}{\zeta_n^b} \cdot \mathbf{h}_1(\zeta_n, \boldsymbol{\omega}_n) \right| < \frac{1}{2}.$$

Then, for points in  $\Omega$ , the  $b^n$ -th root is defined by taking the branch cut along the negative real axis. Moreover, this condition will also imply convergence of the infinite product (3) on  $\Omega$ , since the corresponding sum of logarithms converges:

$$\sum_{n=1}^{\infty} \log \left| \frac{z_{n+1}}{z_n^b} \right|^{\frac{1}{b^{n+1}}} \leq \sum_{n=1}^{\infty} \frac{1}{b^{n+1}} \log 2.$$

To construct  $\Omega$ , first note that for any  $K_1 > 0$  sufficiently small,  $\{|\mathbf{w}| \leq K_1\} \cap (\mathcal{W}^s(\eta_1) \cup \mathcal{W}_{\text{loc}}^s(S))$  is a compact subset of  $\mathbb{C}^n$ . Since  $\mathbf{g}_1$  is holomorphic on  $\mathbb{C}^n$ , there is a bound  $|\mathbf{g}_1(z, \mathbf{w})| \leq K_2$  on any such compact set. A similar bound holds in the other coordinate system. Therefore, it suffices to show:

**Lemma 7.** *Given any  $K > 0$ , there exists a forward invariant neighborhood of  $S$  in which*

$$(8) \quad \frac{|\mathbf{w}|}{|z|^b} < K \quad \text{and} \quad \frac{|\omega|}{|\zeta|^b} < K.$$

*Proof.* We will take an inductive sequence of  $b$  point blow-ups at each of the two fixed points  $\eta_1$  and  $\eta_2$ . Using the forms of  $f$  given by Lemma 3, the calculation will be the same at each of these two points, so we'll focus on  $\eta_1$ , which is given by  $(z, \mathbf{w}) = (0, \mathbf{0})$ .

We first do a point blow-up at  $\eta_1$ , producing an exceptional divisor  $E_{\eta_1,1}$ . Let  $\tilde{L}_1$  be the proper transform of  $L$ . We then blow-up the point intersection point between  $E_{\eta_1,1}$  and  $\tilde{L}_1$ , producing a new exceptional divisor  $E_{\eta_1,2}$  and proper transform  $\tilde{L}_2$ . We inductively do this  $b-2$  additional times, each time blowing up the intersection point between the previous exceptional divisor and proper transform of  $L$ .

Consider the system of coordinates  $z, \boldsymbol{\lambda} = \frac{\mathbf{w}}{z^b}$  centered at the intersection point of  $E_{\eta_1,b}$  with  $\tilde{L}_b$ . Let us denote  $(z', \boldsymbol{\lambda}') = \tilde{f}(z, \boldsymbol{\lambda})$ , where  $\tilde{f}$  is the extension of  $f$  to the final blow-up. We have

$$\begin{aligned} z' &= z^b + z^b \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda}) \\ \boldsymbol{\lambda}' &= \frac{\mathbf{w}'}{(z')^d} = \frac{\sum_{|c|=a} (z^b \boldsymbol{\lambda})^c \mathbf{g}_c(z, z^b \boldsymbol{\lambda})}{(z^b + z^b \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda}))^b} = \frac{z^{b(a-b)} \sum_{|c|=a} \boldsymbol{\lambda}^c \mathbf{g}_c(z, z^b \boldsymbol{\lambda})}{(1 + \boldsymbol{\lambda} \cdot \mathbf{g}(z, z^b \boldsymbol{\lambda}))^b}. \end{aligned}$$

Notice that this extension  $\tilde{f}$  is holomorphic in a neighborhood of  $(z, \boldsymbol{\lambda}) = (0, \mathbf{0})$  and that this point is superattracting for  $\tilde{f}$ .

Therefore, for any  $\varepsilon_1 > 0$  and  $K \geq \delta_1 > 0$ , sufficiently small,  $\tilde{U}_1 := \{|z| < \varepsilon_1, |\boldsymbol{\lambda}| < \delta_1\}$  will be forward invariant under  $\tilde{f}$ . Hence,

$$U_1(\varepsilon_1, \delta_1) := \pi \left( \tilde{U}_1(\varepsilon_1, \delta_1) \right) = \left\{ |z| < \varepsilon_1, \frac{|\mathbf{w}|}{|z|^b} < \delta_1 \right\}$$

will be a forward invariant set for  $f$ .

As stated before, the same calculation can be done at  $\eta_2$ , with analogous results. In particular, for any  $\varepsilon_2 > 0$  and  $K \geq \delta_2 > 0$  sufficiently small we will have a forward invariant set for  $f$  of the form

$$U_2(\varepsilon_2, \delta_2) = \left\{ |\zeta| < \varepsilon_2, \frac{|\omega|}{|\zeta|^b} < \delta_2 \right\}.$$

Let  $V \subset N$  be a forward invariant tubular neighborhood of  $L$  and let

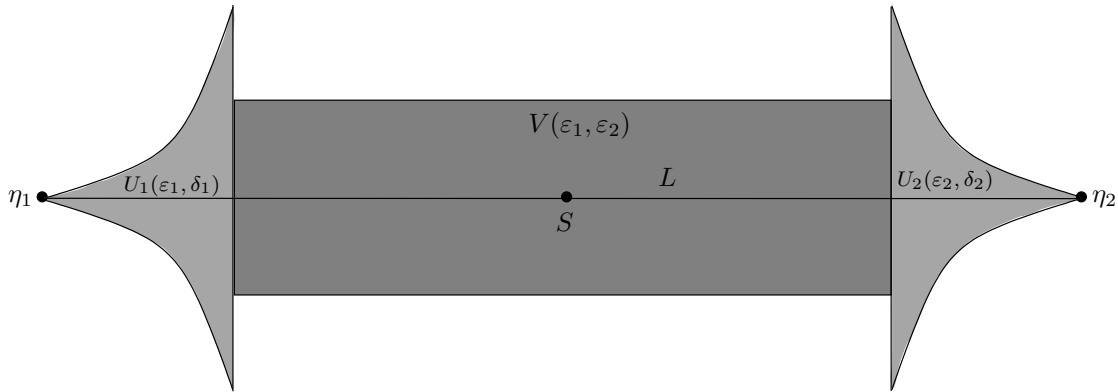
$$V(\varepsilon_1, \varepsilon_2) = V \setminus (\{|z| < \varepsilon_1\} \cup \{|\zeta| < \varepsilon_2\}).$$

Note that if  $V$  sufficiently small, then all points of  $V(\varepsilon_1, \varepsilon_2)$  satisfy (8). We will show that  $V$  can be made even smaller, if necessary, in order to make

$$\Omega := V(\varepsilon_1, \varepsilon_2) \cup U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$$

forward invariant.

Since  $U_1(\varepsilon_1, \delta_1)$  and  $U_2(\varepsilon_2, \delta_2)$  are forward invariant, we need only check that if  $x \in V(\varepsilon_1, \varepsilon_2)$  and  $f(x) \notin V(\varepsilon_1, \varepsilon_2)$ , then  $f(x) \in U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$ . Let us focus on  $x \in \mathcal{W}^s(\eta_1)$ , since the proof will be the same for  $x \in \mathcal{W}^s(\eta_2)$ .

FIGURE 1. The forward invariant neighborhood  $\Omega$ 

Let  $x = (z, \mathbf{w}) \in V(\varepsilon_1, \varepsilon_2) \cap \mathcal{W}^s(\eta_1)$  and let  $(z_1, \mathbf{w}_1) = f(z, \mathbf{w})$ . Since  $(z, \mathbf{w}) \in V(\varepsilon_1, \varepsilon_2)$ ,  $|\mathbf{w}|/|z|^b < K$ , so that (6) and (4) imply that the  $|z_1| \geq |z|^b/2 \geq \varepsilon_1^b/2$ . Thus, we need only choose the (forward invariant) tubular neighborhood  $V$  sufficiently small so that

$$V \cap \left\{ \frac{\varepsilon_1^b}{2} \leq |z| \leq \varepsilon_1 \right\} \subset U_1(\varepsilon_1, \delta_1).$$

Doing the same thing near  $\eta_2$ , we construct a forward invariant neighborhood  $\Omega$  satisfying (8).  $\square$

**2.3. Completing the proof of Theorem 1'.** Using the invariance (2), for any  $(z, \mathbf{w}) \in \mathcal{W}_{\text{loc}}^s(S)$  we have  $|\varphi(z, \mathbf{w})| = 1$  so that  $\psi := \log |\varphi|$  will be a real analytic function that vanishes on  $\mathcal{W}_{\text{loc}}^s(S)$ . Notice on that  $L$ , we have  $\varphi(z, \mathbf{0}) = z$  and hence  $\psi(z, \mathbf{0}) = \log |z|$ . Since the derivative  $D\psi$  is non-zero on  $S$ , we have that  $\{\psi = 0\}$  is a real analytic  $2n - 1$  real-dimensional manifold in some neighborhood of  $S$ .

By Corollary 6,  $\mathcal{W}^s(S) \subset \{\psi = 0\}$  is also a real  $2n - 1$  dimensional manifold. Thus, by invariance of domain,  $\mathcal{W}^s(S) = \{\psi = 0\}$  in this neighborhood.  $\square$ (Theorem 1')

### 3. EXAMPLES ILLUSTRATING THEOREM 1

**3.1. Degenerate Newton Mappings.** Newton mappings used to find the common roots of  $P(x, y) = x(1 - x)$  and  $Q(x, y) = y^2 + Bxy - y$  were considered dynamically in [24]. They have the form

$$(9) \quad N(x, y) = \left( \frac{x^2}{2x - 1}, \frac{y(Bx^2 + 2xy - Bx - y)}{(2x - 1)(Bx + 2y - 1)} \right).$$

We will consider their extension as rational maps of  $\mathbb{P}^1 \times \mathbb{P}^1$ . They are skew products with the first coordinate having superattracting fixed points of degree 2 at  $x = 0$  and  $x = 1$ , so the vertical lines  $\{x = 0\} \times \mathbb{P}^1$  and  $\{x = 1\} \times \mathbb{P}^1$  are transversally superattracting for  $N$  with the same degree. Using the formula, one can check that  $N$  has no indeterminate points in some neighborhood of these two lines.

Restricted to  $\{x = 0\} \times \mathbb{P}^1$ ,  $N$  is the one-dimensional Newton map for the quadratic polynomial with roots at  $y = 0$  and  $y = 1$ . It is therefore conjugate to  $z \mapsto z^2$ , having an invariant circle  $S_0$  corresponding to the points of equal distance from  $y = 0$  and  $y = 1$  in  $\mathbb{P}^1$ . ( $S_0$  is the closure of  $\text{Im}(y) = \frac{1}{2}$  in  $\mathbb{P}^1$ .)

Similarly, the restriction of  $N$  to  $\{x = 1\} \times \mathbb{P}^1$  is the one-dimensional Newton map for the quadratic polynomial with roots at  $y = 0$  and  $y = 1 - B$ . Thus, it is conjugate to  $z \mapsto z^2$ , with an

invariant circle  $S_1$  corresponding to the points of equal distance from  $y = 0$  and  $y = 1 - B$  within  $\mathbb{P}^1$ .

Both of the lines  $\{0\} \times \mathbb{P}^1$  and  $\{1\} \times \mathbb{P}^1$  is transversally superattracting with degree 2, with the restriction of  $N$  to each of them conjugate to  $z \mapsto z^2$ . Therefore, it follows immediately from Theorem 1 that the local stable manifolds  $\mathcal{W}_{\text{loc}}^s(S_0)$  and  $\mathcal{W}_{\text{loc}}^s(S_1)$  are real analytic. This was proven previously in [24] using more specific details of the mapping.

**3.2. Polynomial Skew Products.** For any  $a \in \mathbb{C}$ , the polynomial skew product

$$f_a(z, w) = (z^2, w^2 + az)$$

extends as a holomorphic endomorphism of  $\mathbb{P}^2$ . They were considered in [8] and [11]. The projective vertical line  $\{z = 0\}$  is transversally superattracting with degree 2 and, restricted to it, we have  $f_a(0, w) = w^2$ . Therefore, the fiber-wise Julia set  $J_{z=0} = \{z = 0, |w| = 1\}$  is an invariant circle. It follows from Theorem 1 that  $\mathcal{W}_{\text{loc}}^s(J_{z=0})$  is real analytic. (This fact was stated and used in the proof of Theorem 6.3 from [11].)

**3.3. Example with indeterminacy.** Consider the polynomial mapping  $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  given by

$$(10) \quad g(x, y) = (x^2 + y(1 + xy), y^3(1 + xy)).$$

Within  $\mathbb{C}^2$ , the line  $L := \{y = 0\}$  is invariant and transversally superattracting with degree 3 and  $g|_L$  is given by  $x \mapsto x^2$ . Let  $S := \{|x| = 1, y = 0\}$  be the invariant circle. Although there is the needed domination between the degrees ( $3 > 2$ ), to apply Theorem 1 we need to check how  $g$  extends to a neighborhood of infinity on  $L$ . The extension of  $g$  to  $\mathbb{P}^2$  is given in homogeneous coordinates by

$$g[X : Y : Z] = [X^2 Z^3 + Y Z^2 (Z^2 + XY) : Y^3 (Z^2 + XY) : Z^5].$$

There is a point of indeterminacy for  $g$  at  $[1 : 0 : 0]$  on the projective line  $Y = 0$ , which we'll also denote by  $L$ . Therefore, Theorem 1 does not immediately apply.

Let us perform two blowups. We first blow-up the point  $[1 : 0 : 0]$  and we then blow-up the point where the proper transform of  $L$  intersects the exceptional divisor over  $[1 : 0 : 0]$ . We'll denote the space obtained after doing these two blow-ups by  $\widetilde{\mathbb{P}^2}$ , the projection by  $\pi : \widetilde{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ , the proper transform of  $L$  after these two blow-ups by  $\widetilde{L}$ , the invariant circle within  $\widetilde{L}$  by  $\widetilde{S}$ , and the lift of  $g$  to the blown-up space by  $\widetilde{g} : \widetilde{\mathbb{P}^2} \rightarrow \widetilde{\mathbb{P}^2}$ .

A neighborhood of  $\widetilde{L}$  can be described by two systems of coordinates  $(x, y)$  and  $(\zeta, \tau)$ , where  $x = X/Z, y = Y/Z$  are the original affine coordinates on  $\mathbb{C}^2$  and  $\zeta = Z/X, \tau = XY/Z^2$ . In the first system of coordinates,  $\widetilde{g}$  is given by (10). In the second system of coordinates,  $\widetilde{g}$  is given by

$$\widetilde{g}(\zeta, \tau) = \left( \frac{\zeta^2}{1 + \tau \zeta^3 (1 + \tau)}, \tau^3 \zeta (1 + \tau) (1 + \tau \zeta^3 + \tau^2 \zeta^3) \right).$$

In the second system of coordinates,  $\widetilde{L}$  is given by  $\tau = 0$ , so we see that  $\widetilde{g}$  is holomorphic in a neighborhood of  $\widetilde{L}$ . Moreover,  $\widetilde{L}$  is invariant and transversally superattracting with degree 3 and  $\widetilde{g}|_{\widetilde{L}}$  still given by  $x \mapsto x^2$ . Therefore, Theorem 1 applies to give that the local stable manifold  $\mathcal{W}_{\text{loc}}^s(\widetilde{S})$  for  $\widetilde{S}$  under  $\widetilde{g}$  is real analytic.

Notice that  $\widetilde{g}$  and  $g$  are birationally conjugate by means of  $\pi$ . Moreover, restricted to small neighborhoods of  $\widetilde{S}$  and  $S$ , this birational conjugacy becomes an honest holomorphic conjugacy. Since the local stable manifolds  $\mathcal{W}_{\text{loc}}^s(\widetilde{S})$  and  $\mathcal{W}_{\text{loc}}^s(S)$  are defined in terms of the action of iterates of  $\widetilde{g}$  and  $g$ , respectively, on these small neighborhoods, we conclude that  $\mathcal{W}_{\text{loc}}^s(S)$  is also real analytic.

*Remark.* This third example illustrates two important considerations about Theorem 1. First, it illustrates that one sometimes needs to do some blow-ups in order to obtain a map without indeterminacy in a neighborhood of  $L$ .

Second, it illustrates the reason why we need to consider arbitrary  $\mathbb{C}^{n-1}$  bundles over  $L$ , since blowing up points of  $L$  will change its normal bundle. (In this example, the normal bundle of  $L$  is the hyperplane bundle, while the normal bundle of  $\tilde{L}$  is the tautological bundle.)

#### 4. EXAMPLES FOR WHICH $\mathcal{W}_{\text{loc}}^s(S)$ IS NOT REAL ANALYTIC

We'll now show that the hypothesis in Theorem 1 that  $L$  is transversally superattracting with degree greater than or equal to the degree of  $f|_L$  cannot be eliminated without adding additional hypotheses.

The Migdal-Kadanoff Renormalization map  $R : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  for the Ising model on the DHL is given in homogeneous coordinates by

$$R[U : V : W] = [(U^2 + V^2)^2 : V^2(U + W)^2 : (W^2 + V^2)^2].$$

For this map, the projective line  $L_0 = \{V = 0\}$  is transversally superattracting with degree 2 with  $R$  holomorphic on a forward invariant neighborhood of  $L_0$ . Restricted to  $L_0$ ,  $R$  is given by  $u \mapsto u^4$ , where  $u = U/W$ , so  $a = 2$  and  $b = 4$ . The invariant circle is denoted  $B := \{V = 0, |u| = 1\}$ . Below, we will show that  $\mathcal{W}_{\text{loc}}^s(B)$  is not real analytic in the neighborhood of any point of  $B$ , thus proving Theorem 2.

The second example for which  $a < b$  and  $\mathcal{W}^s(S)$  is not real analytic is the following polynomial skew product of  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  given in affine coordinates by

$$f(z, w) = (z^3 + 2w^2z, w^2).$$

One can check that this map is holomorphic on a forward invariant neighborhood in  $\mathbb{P}^2$  of the invariant line  $L = \{w = 0\}$ . Moreover,  $L$  is transversally superattracting with degree 2, and  $f|_L$  is given by  $z \mapsto z^3$ . Thus,  $a = 2 < 3 = b$ . For this map,  $\mathcal{W}_{\text{loc}}^s(S)$  is not real analytic in the neighborhood of any point of  $S$ .

In this section, we'll provide a detailed proof of Theorem 2, showing that  $\mathcal{W}_{\text{loc}}^s(B)$  is not real analytic. An adaptation of the same techniques can be used to show the analogous result for the skew product  $f$ . We leave details of this adaptation to the reader.

**4.1. The Migdal-Kadanoff Renormalization.** In the remainder of this section, we will adopt the notation from the recent preprints [5, 4] by Bleher, Lyubich, and Roeder. Although  $R : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is more convenient for illustrating Theorem 1, in the proof of Theorem 2 it will be more convenient to work the expression of the Migdal-Kadanoff renormalization  $\mathcal{R} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  in the physical coordinates  $(z, t)$ . In these coordinates, it is given by

$$(11) \quad (z_{n+1}, t_{n+1}) = \left( \frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2}, \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}} \right) := \mathcal{R}(z_n, t_n).$$

We consider  $(z, t)$  as affine coordinates on  $\mathbb{P}^2$  with  $z = Z/Y, t = T/Y$  for some system of homogeneous coordinates  $[Z : T : Y]$ . The map  $\mathcal{R}$  has an invariant projective line  $\mathcal{L}_0 = \{T = 0\}$  that is transversally superattracting, except for an indeterminate point at  $\mathbf{0} := [0 : 0 : 1]$ , and  $\mathcal{R}|_{\mathcal{L}_0}$  is given by  $z \mapsto z^4$ . The invariant circle is given by  $\mathcal{B} = \{|z| = 1, t = 0\}$ .

The map  $R$  is semi-conjugate to  $\mathcal{R}$  by means of a rational map  $\Psi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ :

$$(12) \quad \begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\mathcal{R}} & \mathbb{P}^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{P}^2 & \xrightarrow{R} & \mathbb{P}^2 \end{array}$$

with  $[U : V : W] = \Psi([Z : T : Y]) = [Y^2 : ZT : Z^2]$ . The map  $\Psi$  sends  $\mathcal{L}_0$  to  $L_0$ ,  $\mathcal{B}$  to  $B$ , and is holomorphic in a neighborhood of  $\mathcal{B}$ . Therefore,  $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \Psi^{-1}(\mathcal{W}_{\text{loc}}^s(B))$ . In particular, if  $\mathcal{W}_{\text{loc}}^s(B)$  were real analytic in the neighborhood of any point of  $B$ , then  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  would be real analytic in the neighborhood of the preimage of that point under  $\Psi$ . So, Theorem 2 will follow from:

**Theorem 2'.** *The stable manifold  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is not real analytic at any point.*

*Remark.* The reason we originally stated Theorem 2 for  $R$  rather than  $\mathcal{R}$  is that  $R$  is holomorphic in a full neighborhood of  $L_0$ , so that it illustrates why the hypothesis on  $a$  and  $b$  can't be eliminated in Theorem 1. One can also resolve the indeterminacy  $\mathbf{0} \in \mathcal{L}_0$  for  $\mathcal{R}$ , placing it in the context of Theorem 1, via a suitable birational modification (two blow-ups and one blow-down), but that is somewhat more complicated.

We will begin by proving the following proposition, and Theorem 4.1 will follow shortly thereafter.

**Proposition 8.**  *$\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is not real analytic in any full neighborhood of  $\mathcal{B}$ .*

We will begin by describing the dynamics of  $\mathcal{R}$  near  $\mathcal{L}_0$ , and after that, the proof divides into four main parts: construction of a Böttcher function  $\varphi$ , development of local properties of  $\mathcal{R}$  near the fixed point  $\eta'$ , the extension of the domain of  $\varphi$ , and the behavior of  $\varphi$  and  $\mathcal{R}$  in the extension, which concludes with a proof of Proposition 8.

**4.2. Dynamics in a Neighborhood of  $\mathcal{L}_0$ .** We will now briefly summarize basic properties of the dynamics for  $\mathcal{R}$  in a neighborhood of  $\mathcal{L}_0$  from [5, Section 4].

Let  $\mathbb{D}_0 := \{|z| < 1, t = 0\} \subset \mathcal{L}_0$ . The orbit of any  $z \in \mathbb{D}_0$  will converge to an indeterminate point  $\mathbf{0} := \{(0, 0)\}$ . (Informally, we will denote these points by  $\mathcal{W}^s(\mathbf{0})$ .) Meanwhile, points near  $\mathbf{0}$  but not on  $\mathcal{L}_0$  will converge to a superattracting fixed point  $\eta := \{(0, 1)\}$ .

To see what happens for large  $|z|$ , we write  $\mathcal{R}$  in homogeneous coordinates, obtaining

$$(13) \quad \mathcal{R}: [Z : T : Y] \mapsto [Z^2(Z^2 + T^2)^2 : T^2(Z^2 + Y^2)^2 : (Z^2 + T^2)(T^2Z^2 + Y^4)].$$

There is another superattracting fixed point  $\eta' := [1 : 0 : 0]$ , which attracts all points of  $\mathcal{L}_0$  with  $|z| > 1$ .

**Lemma 9.**  *$\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}_{\text{loc}}^s(\eta) \cup \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cup \mathcal{W}_{\text{loc}}^s(\eta')$  fills some neighborhood of  $\mathcal{L}_0 \setminus \{\mathbf{0}\}$ .*

See [5, Lemma 4.2].

There is another invariant line  $\mathcal{L}_1 := \{t = 1\}$  passing through  $\eta$  and  $\eta'$ . We have  $\mathcal{R}|_{\mathcal{L}_1}: z \rightarrow z^2$ .

For the remainder of this section, it will be convenient to use a system of affine coordinates centered at  $\eta'$ . We will use  $(\lambda = Y/Z - T/Z, \tau = T/Z)$ , so that  $\mathcal{L}_0 = \{\tau = 0\}$  and  $\mathcal{L}_1 = \{\lambda = 0\}$ . In these coordinates,

$$(14) \quad (\lambda_{n+1}, \tau_{n+1}) = \left( \lambda_n^2 \left( \frac{\lambda_n + 2\tau_n}{1 + \tau_n^2} \right)^2, \tau_n^2 \left( \frac{1 + (\tau_n + \lambda_n)^2}{1 + \tau_n^2} \right)^2 \right) := \mathcal{R}(\lambda_n, \tau_n).$$

As before,  $\mathcal{R}|_{\mathcal{L}_0}: \lambda \rightarrow \lambda^4$  and  $\mathcal{R}|_{\mathcal{L}_1}: \tau \rightarrow \tau^2$ .

**4.3. Böttcher Function  $\varphi$ .** We continue by exploring some preliminary consequences of the hypothesis that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic in such a full neighborhood of  $\mathcal{B}$ .

**Proposition 10.** *If  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic in a full neighborhood of  $\mathcal{B}$ , then there is another neighborhood  $\Omega_0$  of  $\mathcal{B}$  and a holomorphic function  $\varphi: \Omega_0 \rightarrow \mathbb{C}$  such that*

- (i) if  $(\lambda, \tau) \in \Omega_0$  and  $\mathcal{R}(\lambda, \tau) \in \Omega_0$ , then  $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$ ,
- (ii)  $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$ , and
- (iii)  $\varphi(\lambda, 0) = \lambda$ .

*Remark.* The function  $\varphi$  is analogous to the one constructed in the Proof of Theorem 1. However, Proposition 10 only gives that  $\varphi$  is defined on a small neighborhood of  $\mathcal{B}$ , which may not be forward invariant under  $\mathcal{R}$ .

We will exploit the fact that each  $x \in B$  is hyperbolic, emitting a stable manifold  $\mathcal{W}_{\text{loc}}^s(x)$  that is a one-dimensional holomorphic curve transverse to  $\mathcal{L}_0$ . Together, the union of stable manifolds of each  $x \in B$  forms a foliation of  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ ; see [5, Proposition 9.2].

The notion of Levi-flat real-codimension 1 hypersurfaces  $\Sigma \subset \mathbb{C}^n$  will be useful; for background see [14, 19]. A  $C^2$  hypersurface  $\Sigma$  is Levi flat if through each point of  $\Sigma$  there is a complex codimension 1, holomorphic hypersurface. The union of these hypersurfaces is called the *Levi foliation* of  $\Sigma$ . Thus, the preceding paragraph shows that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is Levi flat. Note that there is another, more common but equivalent, definition of Levi-flat given in terms of vanishing an appropriate Levi  $(1, 1)$ -form [14, page 126].

Rea's Theorem [23] holds in any codimension, but here we need only

**Rea's Theorem in Codimension 1.** *Suppose  $\Sigma$  is a Levi-flat, real analytic hypersurface defined on some open  $\Omega_0 \subset \mathbb{C}^n$ . Then there is a neighborhood  $\Omega \subset \Omega_0$  of  $\Sigma$  to which the Levi foliation extends uniquely and holomorphically.*

We omit the proof, as it is rather simple in this case.

*Proof of Proposition 10.* As stated above,  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is foliated by a family  $\mathcal{F}$  of holomorphic stable curves at each point in  $\mathcal{B}$ , so it's Levi flat. Since  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is assumed to be real analytic, Rea's Theorem implies that this Levi foliation extends to be a complex analytic foliation in a neighborhood of  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ . Since the foliation  $\mathcal{F}$  is transverse to  $\mathcal{L}_0$  at points of  $\mathcal{B}$ , in a small enough neighborhood  $\tilde{\Omega}$ , each curve  $\gamma_x$  of the foliation is transverse to  $\mathcal{L}_0$ . Then we may assume  $\Omega$  is the union of connected components in  $\tilde{\Omega}$  of any leaf that intersects  $\tilde{\Omega} \cap \{\lambda = 0\}$ . Let  $\varphi: \Omega \rightarrow \mathbb{C}$  be the map assigning to each  $(\lambda, \tau) \in \Omega$  the point where  $\gamma_{(\lambda, \tau)}$  intersects  $\tau = 0$ . From this, (ii) and (iii) follow immediately. Note that it follows from a change of coordinates and the Implicit Function Theorem that  $\varphi$  is holomorphic.

Define  $\Omega_0$  to be the connected component of  $\mathcal{R}^{-1}(\Omega) \cap \Omega$  containing  $\mathcal{B}$ . For each  $\tau_0$  with  $|\tau_0|$  sufficiently small, let  $\mathcal{L}_{\tau_0} := \{\tau = \tau_0\}$ . Observe that  $\mathcal{B}_{\tau_0} := \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cap \mathcal{L}_{\tau_0}$  is a topological circle. Since  $\mathcal{B}_{\tau_0} \subset \mathcal{W}_{\text{loc}}^s(\mathcal{B})$ , (i) holds on  $\mathcal{B}_{\tau_0}$  and, by uniqueness properties of holomorphic functions, it holds in some open neighborhood of  $\mathcal{B}_{\tau_0}$  within  $\mathcal{L}_{\tau_0}$ . Varying  $\tau_0$ , these neighborhoods form an open neighborhood of  $\mathcal{B}$  contained in  $\Omega_0$  on which (i) holds. This property then extends to all of  $\Omega_0$ , since  $\Omega_0$  is connected.  $\square$

We can suppose that the domain  $\Omega_0$  on which  $\varphi$  is defined, given by Proposition 10, is sufficiently small, so that it is contained in  $\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}_{\text{loc}}^s(\eta) \cup \mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cup \mathcal{W}_{\text{loc}}^s(\eta')$ . Since  $\mathcal{B}$  has a local product structure, it is isolated in the recurrent set. Proof of this is similar to [22, Proposition 4.4]. Thus, we can choose  $\Omega_0$  smaller if necessary so that each orbit enters and leaves  $\Omega_0$  at most once.

**Proposition 11.** *The domain  $\Omega_0$  may be extended to  $\Omega$ , a neighborhood of  $\mathcal{L}_0 \setminus \{\eta', \eta\}$ , such that  $\varphi: \Omega \rightarrow \mathbb{C}$  is holomorphic,*

- (i) *If  $(\lambda, \tau) \in \Omega$  and  $\mathcal{R}(\lambda, \tau) \in \Omega$ , then  $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$ ,*
- (ii)  *$\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$ , and*
- (iii)  *$\varphi(\lambda, 0) = \lambda$  for  $x \in \mathcal{L}_0 \setminus \{\eta', \eta\}$ .*

In general, the push-forward of a function by a mapping is not well-defined. However, if the mapping is proper, then it is well-defined by averaging over the fibers. It was shown in [5, Sec. 4.5] that  $\mathcal{R}$  has topological degree eight. In the proposition below, we mimic this push forward under a proper mapping.

*Proof.* Let  $\Omega_n := \{x : \mathcal{R}^{-n}\{x\} \subseteq \Omega_0\}$  and  $C_n$  be the critical value set for  $\mathcal{R}^n$ . For  $x \in \Omega_n \setminus C_n$ , we may define

$$(15) \quad \varphi(x) = \frac{1}{8^n} \sum_{i=1}^{8^n} \varphi(y_i)^4,$$

where  $\{y_i\}_{i=1}^{8^n} = \mathcal{R}^{-n}\{x\}$ . Then locally about each  $x \in \Omega_n \setminus C_n$ ,  $\varphi$  is holomorphic since each branch of  $\mathcal{R}^{-n}$  is holomorphic by the Inverse Function Theorem. If  $x$  follows a nontrivial loop around  $C_n$ , then  $\varphi(x)$  has no monodromy since we are averaging over all of the fibers in (15). Moreover, since  $|\varphi|$  is bounded on  $\Omega_0$ , (15) implies  $|\varphi|$  is also bounded on  $\Omega_n \setminus C_n$ . Therefore, by the Riemann Extension Theorem,  $\varphi$  can be extended through the critical value curves to be holomorphic on all of  $\Omega_n$ .

If  $x \in \Omega_n \cap \Omega_m$  with  $n \geq m \geq 0$ , then  $\mathcal{R}^{-n}\{x\}, \mathcal{R}^{-m}\{x\} \subset \Omega_0$ . Since any orbit enters and leaves  $\Omega_0$  at most once, for any  $y_i \in \mathcal{R}^{-m}\{x\}$  and each  $z_j \in \mathcal{R}^{m-n}\{y_i\}$ , we have that  $z_j, \mathcal{R}(z_j), \dots, \mathcal{R}^{n-m}(z_j) = y_i \in \Omega$ . Thus,  $\varphi(y_i) = \varphi(\mathcal{R}^{n-m}(z_j)) = \varphi(z_j)^{4^{n-m}}$  since (i) holds on  $\Omega_0$ . This implies

$$\frac{1}{8^m} \sum_{y_i \in \mathcal{R}^{-m}(x)} \varphi(y_i)^{4^m} = \frac{1}{8^n} \sum_{z_j \in \mathcal{R}^{-n}(x)} \varphi(z_j)^{4^n},$$

so that the two definition of  $\varphi$  agree in  $\Omega_n \cap \Omega_m$ .

We obtain a well-defined holomorphic function  $\varphi$  on

$$(16) \quad \Omega_\infty := \bigcup_{n=0}^{\infty} \Omega_n.$$

Then we define  $\Omega$  to be the connected component of  $\mathcal{R}^{-1}(\Omega_\infty) \cap \Omega_\infty$  containing  $\mathcal{B}$ . Now (i) holds on all of  $\Omega$  using the exactly the same proof as in Proposition 10.i.

Since  $\mathcal{L}_0$  is forward invariant,  $\Omega_0$  intersects  $\mathcal{L}_0$ , and  $\mathcal{R}|_{\mathcal{L}_0}$  is  $\lambda \mapsto \lambda^4$ , it follows that  $\Omega$  contains  $\mathcal{L}_0 \setminus \{\eta', \eta\}$ . The fact that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$  also follows from the fact that  $\Omega_0 \subset \Omega$ .  $\square$

**4.4. Local Properties Near  $\eta'$ .** In order to study the geometry of the extended domain  $\Omega$  and the properties of  $\varphi$ , several technical results about the dynamics near  $\eta'$  will be required. We may choose  $\varepsilon > 0$  sufficiently small so that the bidisk

$$(17) \quad X_\varepsilon := \{|\lambda| < \varepsilon, |\tau| < \varepsilon\},$$

is forward invariant, and  $\mathcal{R}$  strictly decreases each component in modulus. We continue by describing the trajectory of orbits as they converge to  $\eta'$ .

**Proposition 12.** *If  $\varepsilon > 0$  is sufficiently small, then for any  $\gamma \in \mathbb{Z}_+$ , if  $(\lambda_0, \tau_0) \in X_\varepsilon \setminus \mathcal{L}_0$ , then  $|\lambda_n|/|\tau_n|^\gamma \rightarrow 0$ .*

This proposition implies that any point near  $\eta'$  and not on  $\mathcal{L}_0$  converges to  $\eta'$  with an arbitrarily high degree of tangency to  $\mathcal{L}_1$ .

*Proof.* We first proof the proposition when  $|\lambda_0| \leq |\tau_0|^\gamma$ . Let  $w_n := \lambda_n/\tau_n^\gamma$ , so that

$$(18) \quad w_{n+1} = \frac{\lambda_{n+1}}{\tau_{n+1}^\gamma} = \frac{\lambda_n^2}{\tau_n^{2\gamma}} \left( \frac{(1 + \tau_n^2)^{\gamma-1} (\lambda_n + 2\tau_n)}{(1 + (\tau_n + \lambda_n)^2)^\gamma} \right)^2 = w_n^2 \tau_n^2 \left( \frac{(1 + \tau_n^2)^{\gamma-1} (2 + w_n \tau_n^{\gamma-1})}{(1 + \tau_n (1 + w_n \tau_n^\gamma)^2)^\gamma} \right)^2.$$

In the  $(\tau, w)$  coordinates,  $(0, 0)$  is a superattracting fixed point for  $\mathcal{R}$ . Then there is a  $\delta > 0$  such that any point with  $|\tau|, |w| < \delta$  is in  $\mathcal{W}^s((0, 0))$ . The closed disk  $\{\tau = 0, |w| \leq 1\}$  collapses to  $(0, 0)$ . By continuity, there exists  $\varepsilon > 0$  such that

$$(19) \quad \mathcal{R}(\{|\tau| < \varepsilon, |w| \leq 1 + \varepsilon\}) \subset \{|\tau|, |w| < \delta\} \subset \mathcal{W}^s((0, 0)).$$

Thus, for  $(\lambda_0, \tau_0) \in X_\varepsilon$  with  $\varepsilon > 0$  sufficiently small, if  $|\lambda_0| \leq |\tau_0|^\gamma$ , then the result follows.

Now it suffices to show that if  $\tau_0 \neq 0$ , then there is some  $N \geq 0$  so that  $|\lambda_n| \leq |\tau_n|^\gamma$  for any  $n \geq N$ . Let

$$(20) \quad M_1 = \min_{(\lambda, \tau) \in \bar{X}_\varepsilon} \left| \frac{1 + (\tau + \lambda)^2}{1 + \tau^2} \right|^2 \quad \text{and} \quad M_2 = \max_{(\lambda, \tau) \in \bar{X}_\varepsilon} 9 \left| \frac{1}{1 + \tau^2} \right|^2.$$

As long as  $|\lambda_n| \geq |\tau_n|^\gamma$ , we have

$$(21) \quad |\tau_{n+1}| \geq M_1 |\tau_n|^2 \quad \text{and} \quad |\lambda_{n+1}| \leq M_2 |\lambda_n|^{2+2/\gamma}.$$

This implies that

$$(22) \quad |\tau_n| \geq A_1 \rho_1^{2^n} \quad \text{and} \quad |\lambda_n| \leq A_2 \rho_2^{(2+2/\gamma)^n}$$

for some  $A_i > 0$  and  $0 < \rho_i < 1$ . Then

$$(23) \quad \frac{|\lambda_n|}{|\tau_n|^\gamma} \leq \frac{A_2 \rho_2^{(2+2/\gamma)^n}}{A_1 \rho_1^{\gamma 2^n}} = A \rho_2^{(2+2/\gamma)^n - \alpha \gamma 2^n} \rightarrow 0,$$

where  $\rho_1 = \rho_2^\alpha$  and  $A = A_2/A_1$ . Thus, for some iterate  $m$ , we have  $|\lambda_m| \leq |\tau_m|^\gamma$ .  $\square$

Consider the ‘‘bullet-shaped’’ regions  $B_{\gamma, c} := \{(\lambda, \tau) : |\lambda| \geq c|\tau|^\gamma\}$ , and let  $B_\gamma \equiv B_{\gamma, 1}$ .

**Corollary 13.** *If  $\varepsilon > 0$  is sufficiently small, then for any  $\gamma \in \mathbb{Z}_+$ ,  $\mathcal{R}^{-1}(B_\gamma) \cap X_\varepsilon \subset B_\gamma$ .*

**Corollary 14.** *For any  $\gamma \in \mathbb{Z}_+$ ,  $\bigcap_{n=0}^{\infty} \mathcal{R}^{-n}(B_\gamma) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon$*

**Lemma 15.** *For any sufficiently small  $\varepsilon > \sigma > 0$  and any  $\gamma \in \mathbb{Z}_+$ , there exist  $m \in \mathbb{Z}_+$  such that  $\mathcal{R}^{-m}(B_\gamma) \cap (\bar{X}_\varepsilon \setminus X_\sigma) \subset C^h$ .*

*Proof.* Consider the compact set  $K := (\bar{X}_\varepsilon \setminus X_\sigma) \cap C^v$ . It suffices to prove that there exists  $m \in \mathbb{Z}_+$  such that  $\mathcal{R}^m(K) \subset X_\varepsilon \setminus B_\gamma$ . By the proof of Proposition 1.6, for each  $x \in K$ , there exists  $m_x$  such that for any  $m \geq m_x$ ,  $\mathcal{R}^m x \in X_\varepsilon \setminus B_\gamma$ , which is open. Then there is an open neighborhood  $U_x$  of  $x$  such that  $\mathcal{R}^{m_x}(U_x) \subset X_\varepsilon \setminus B_\gamma$ . Since  $K$  is compact, there exists  $m$  such that for any  $x \in K$ ,  $\mathcal{R}^m(x) \in X_\varepsilon \setminus B_\gamma$ .  $\square$

Recall that  $\mathcal{R}|_{\mathcal{L}_0}$  is  $\lambda \mapsto \lambda^4$  and  $\mathcal{R}|_{\mathcal{L}_1}$  is  $\tau \mapsto \tau^2$ . The following distortion estimates allow local approximation of these properties near  $\eta'$ . Let

$$(24) \quad C^v := \{|\tau| \geq |\lambda|\} \quad \text{and} \quad C^h := \{|\tau| \leq |\lambda|\}.$$

Also, recall the notation  $(\lambda_n, \tau_n) = \mathcal{R}^n(\lambda_0, \tau_0)$ . Lastly, given two sequences  $x_n$  and  $y_n$ , we will use  $x_n \asymp y_n$  to mean that  $a \leq |x_n/y_n| \leq A$  for some constants  $0 < a < A$ .

**Proposition 16.** *For  $\varepsilon > 0$  sufficiently small and any  $\gamma \geq 1$ ,*

- (i) *If  $(\lambda_i, \tau_i) \in B_\gamma \cap X_\varepsilon$  for  $i = 0, \dots, n$ , then  $|\lambda_n| \asymp |\lambda_0|^{4^n}$ .*
- (ii) *If  $(\lambda_i, \tau_i) \in X_\varepsilon \setminus B_\gamma$  for  $i = 0, \dots, n$ , then  $|\tau_n| \asymp |\tau_0|^{2^n}$ .*

*Proof.* Let

$$(25) \quad A_i = \frac{1}{|\lambda_{n-i}|^2} \left| \frac{\lambda_{n-i} + 2\tau_{n-i}}{1 + \tau_{n-i}^2} \right|^2 \leq 1 + 5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right|,$$

so that  $|\lambda_{n-i+1}| = A_i |\lambda_{n-i}|^4$ . Inductively, we have

$$(26) \quad |\lambda_n| = \left( \prod_{i=1}^n A_i^{4^{i-1}} \right) |\lambda_0|^{4^n}.$$

Recall the constants  $M_1 \leq 1 \leq M_2$  from the proof of Proposition 12, which are independent of  $\gamma$ . We have  $|\tau_n| \geq (M_1|\tau_{n-i}|)^{2^i}$  and  $|\lambda_n| \leq (M_2|\tau_{n-i}|)^{(2+2/\gamma)^i}$ , so it follows that

$$(27) \quad \left| \frac{\tau_n}{\lambda_n} \right| \geq \frac{M_1^{2^i}}{M_2^{(2+2/\gamma)^i}} \left| \frac{\tau_{n-i}}{\lambda_{n-i}^{(1+1/\gamma)^i}} \right|^{2^i}.$$

This implies there is a  $0 < \delta < 1$  such that

$$(28) \quad 5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right| \leq 5(M_2|\lambda_{n-i}|)^{(1+1/\gamma)^{i-1}} \frac{M_2}{M_1} \left| \frac{\tau_n}{\lambda_n} \right|^{1/2^i} \leq \delta^{(1+1/\gamma)^i},$$

since  $M_2$  is a fixed constant,  $|\lambda_{n-i}| < \varepsilon$ , and we can choose  $\varepsilon$  as small as we like.

It suffices to find uniform constants to estimate the product  $\prod_{i=1}^n A_i^{4^{i-1}}$  independent  $n$ . Observe

$$(29) \quad \prod_{i=1}^n A_i^{4^{i-1}} \leq \prod_{i=1}^n \left( 1 + 5 \left| \frac{\tau_{n-i}}{\lambda_{n-i}} \right| \right)^{4^{i-1}} \leq \prod_{i=1}^{\infty} \left( 1 + \delta^{(1+1/\gamma)^i} \right)^{4^{i-1}},$$

where the last product converges since

$$(30) \quad \sum_{i=1}^{\infty} 4^{i-1} \log \left( 1 + \delta^{(1+1/\gamma)^i} \right)$$

converges. Thus, there is a constant  $A$  such that for any  $n$ ,  $\prod_{i=1}^n A_i^{4^{i-1}} \leq A$ .

A similar calculation can be done to find a uniform lower bound for the product. Moreover, the proof for the vertical distortion control is similar (and easier).  $\square$

Consider  $\mathcal{R}^*(\mathcal{L}_1)$ , the pullback of the curve  $\mathcal{L}_1 = \{T = Y\}$ , given by

$$(31) \quad -Z^2(T - Y)^2(T + Y)^2 = 0.$$

The pullback of  $\mathcal{L}_1$  contains  $\mathcal{L}_1$ ,  $\{Z = 0\}$ , and  $\{T + Y = 0\}$  (each counted with multiplicity two). Call this last curve  $D$ , so in  $(\lambda, \tau)$  coordinates,

$$(32) \quad D := \{\lambda + 2\tau = 0\}.$$

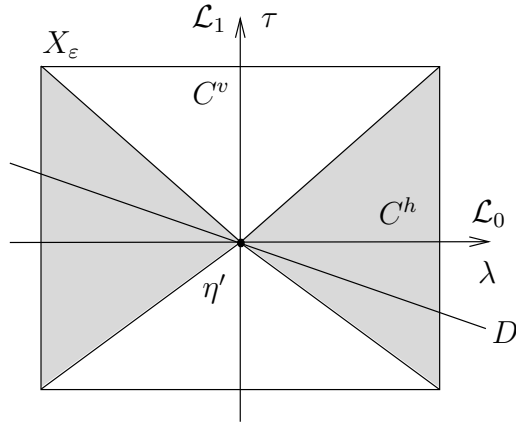


FIGURE 2. Bidisk neighborhood of  $\eta'$

**Lemma 17.** *If  $x \in X_\varepsilon \setminus B_3$  and  $\varepsilon$  is sufficiently small, then  $\mathcal{R}^{-1}\{x\} \cap C^h \neq \emptyset$  and  $\mathcal{R}^{-1}\{x\} \cap C^v \neq \emptyset$ .*

*Proof.* Let  $N := \{|\lambda| < \frac{1}{2}|\tau|^2\}$ , and note that if  $x \in X_\varepsilon \setminus B_3$ , then  $x \in N \cap X_\varepsilon$ . Suppose  $x \in N \cap X_\varepsilon$  and let  $(\lambda, \tau) \in \mathcal{R}^{-1}\{x\}$ . Recall that the line  $D := \{\lambda + 2\tau = 0\}$  has  $\mathcal{R}(D) = \mathcal{L}_1$ . Also, note that  $N$  is the union over  $|c| \leq 1/2$  of the curves  $P_c := \{\lambda = c\tau^2\}$ , and the preimage of any of these curves,  $\mathcal{R}^{-1}(P_c)$ , is the set of points satisfying

$$(33) \quad \lambda^2 \left( \frac{\lambda + 2\tau}{1 + \tau^2} \right)^2 = c\tau^4 \left( \frac{1 + (\lambda + \tau)^2}{1 + \tau^2} \right)^4.$$

It follows that if  $\varepsilon > 0$  is small enough that  $\left| \sqrt{c} \frac{(1 + (\lambda + \tau)^2)^2}{1 + \tau^2} \right| \leq 1$ , then  $\mathcal{R}^{-1}(P_c)$  is a set of points that satisfies

$$(34) \quad \left| \frac{\lambda}{\tau} \right| \frac{|\lambda + 2\tau|}{|\tau|} \leq 1.$$

Since the curve  $P_c$  is tangent to  $\mathcal{L}_1$  and  $\mathcal{R}(D \cup \mathcal{L}_1) = \mathcal{L}_1$ ,  $\mathcal{R}^{-1}(P_c)$  must have a branch tangent to  $\mathcal{L}_1$  and another branch tangent to  $D$ . Moreover, by (34), these preimage curves must be contained in  $C^v$  and  $C^h$  respectively. Thus, there is a preimage in  $C^h$  and another in  $C^v$ .  $\square$

*Remark.* With a small amount of additional work, one can show that any point  $x \in X_\varepsilon$  with  $\varepsilon$  sufficiently small has a preimage under the second iterate of  $\mathcal{R}$  contained in  $C^h \cap X_\varepsilon$ .

**Lemma 18.** *For any sufficiently small  $\varepsilon > 0$  and any  $k \in \mathbb{Z}_+$ , there exist  $\sigma > 0$  and  $\gamma \in \mathbb{Z}_+$  such that if  $x \in X_\sigma \setminus B_\gamma$ , then  $x$  has a preorbit  $\{x_{k,i}^v\}_{i=1}^k$  of length at least  $k$  contained in  $C^v \cap X_\varepsilon$ .*

*Proof.* Let  $\mathcal{R}(\lambda, \tau) = (\lambda', \tau') \in X_\sigma \setminus B_\gamma$ , so there is a  $\delta_1 > 0$  such that

$$(35) \quad 1 \geq \frac{|\lambda'|}{|\tau'|^\gamma} \geq \frac{|\lambda|^2}{|\tau|^{2\gamma}} |\lambda + 2\tau|^2 (1 - \delta_1)^{2(\gamma-1)}.$$

For large enough  $\gamma$  and small enough  $\sigma$ , Lemma 17 implies there is some preimage  $(\lambda, \tau) \in C^v$ . Then  $|\tau| \leq |2\tau + \lambda|$ , so

$$(36) \quad 1 \geq \frac{|\lambda|}{|\tau|^{\gamma-1}} (1 - \delta_1)^{\gamma-1}.$$

There are  $\delta_i$  for  $i = 2, \dots, \gamma-2$  so that after repeating this process, we have  $\mathcal{R}^{\gamma-2}(\lambda_0, \tau_0) \in X_\varepsilon \setminus B_\gamma$  with

$$(37) \quad 1 \geq \frac{|\lambda_0|}{|\tau_0|^3} (1 - \delta_1)^{\frac{\gamma-1}{2\gamma-4}} (1 - \delta_2)^{\frac{\gamma-2}{2\gamma-3}} \dots (1 - \delta_{\gamma-2})^{\frac{4}{2}} (1 - \delta_{\gamma-3})^3.$$

Pick  $\sigma$  small enough and  $\gamma \geq k+3$  so that (37) implies  $(\lambda_0, \tau_0) \in C^v \cap X_\sigma$  and  $\mathcal{R}^{-k}\{x\} \subset X_\varepsilon$ .  $\square$

#### 4.5. Properties of $\Omega$ and $\varphi$ .

**Lemma 19.** *For any  $\gamma \in \mathbb{Z}_+$ , there exists  $\sigma > 0$  such that  $B_\gamma \cap X_\sigma \subset \Omega$ .*

*Proof.* By Proposition 11,  $\Omega$  contains some neighborhood of  $\mathcal{L}_0 \setminus \{\eta', \eta\}$ . By Lemma 16, there exists  $\varepsilon > 0$  sufficiently small so that for any  $\gamma \in \mathbb{Z}_+$ , the horizontal distortion estimates can be applied in  $B_\gamma \cap X_\varepsilon$ . Let  $\mathcal{A} := \{a\varepsilon^{4j+2} < |\lambda| < 2A\varepsilon^{4j}, |\tau| < \delta\}$ , where  $a$  and  $A$  are the constants from the distortion estimate,  $j \in \mathbb{Z}_+$  is chosen so that  $\mathcal{A} \subset X_\varepsilon$ , and  $\delta < 0$  is chosen small enough so that  $\mathcal{A} \subset \Omega$ . See Figure 3.

Let  $x = (\lambda_0, \tau_0) \in B_\gamma \cap X_\sigma$  and  $S_x$  be the real straight line path connecting  $x$  to  $(\lambda_0, 0) \in \mathcal{L}_0$ . If  $\sigma < \varepsilon$  is sufficiently small, then by Corollary 14 and the horizontal distortion estimates, there is an integer  $n$  such that both  $\mathcal{R}^{-n}\{S_x\}, \mathcal{R}^{-n+1}\{S_x\} \subset \mathcal{A}$ . Then  $S_x \subset \Omega_\infty$  and  $S_x \subset \mathcal{R}^{-1}(\Omega_\infty)$ , and since  $S_x$  is connected and intersects  $(\mathcal{L}_0 \setminus \{\eta', \eta\}) \subset \Omega$ , we have that  $x \in S_x \subset \Omega$ .  $\square$

**Proposition 20.** *For any sequence  $\{x_m\} \subset \Omega$ , if  $x_m \rightarrow \eta'$ , then  $\varphi(x_m) \rightarrow 0$ .*

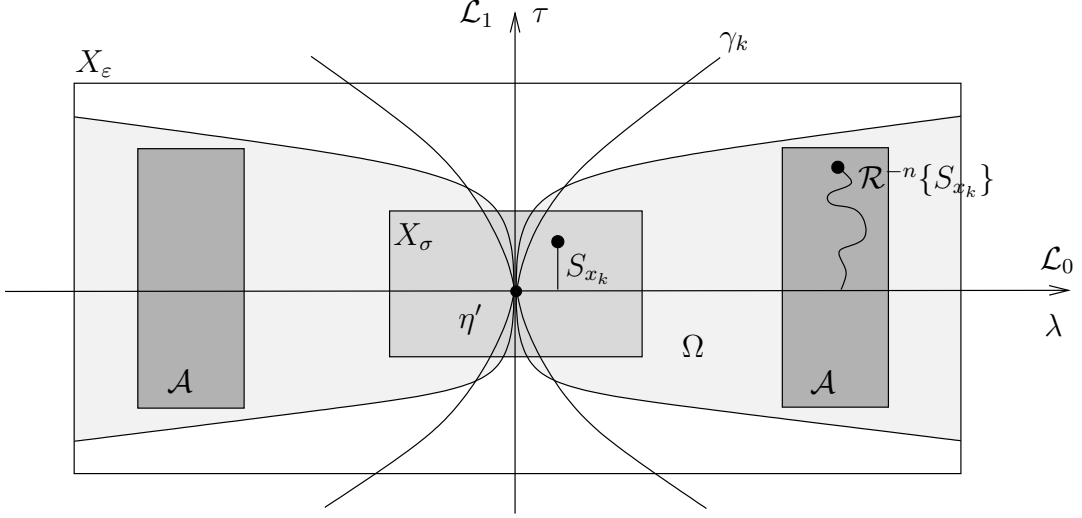


FIGURE 3.  $X_\sigma$  (medium gray),  $\mathcal{A}$  (dark gray), and  $\Omega$  (light gray); proportions have been modified to show detail.

*Proof.* By Lemma 19, there exists  $\sigma > 0$  such that  $B_3 \cap X_\sigma \subset \Omega$ . By the uniformity of  $\varphi$  on compact sets and the fact that  $\varphi|_{\mathcal{L}_0} = id$ , if  $\delta > 0$  small enough, then  $\mathcal{A} := \{\sigma^{4^2} < |\lambda| < \sigma, |\tau| < \delta\} \subset B_3$ , and  $|\varphi(x)| < 2\sigma$  for  $x \in \mathcal{A}$ . By Lemma 17, there is a point in the preimage of each  $x_m \in X_\sigma \setminus B_3$  contained in  $B_3$ , and Corollary 13,  $B_3$  is backward invariant. Thus, there is a backward orbit of each  $x_m$  that remains in  $B_3 \subset \Omega$ . Let  $\{x_{m,n}\}$  be this preorbit. If  $x_m$  sufficiently close to  $\eta'$ , then by Corollary 14 there is an  $N(m)$  such that  $x_{m,N(m)} \in \mathcal{A}$ . Using the invariance  $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$ , we have

$$(38) \quad |\varphi(x_m)| = |\varphi(x_{m,N})^{4^N}| < (2\sigma)^{4^N}.$$

As  $m$  goes to infinity, we need  $N$  to go to infinity as well in order for  $x_{m,N}$  to remain in  $\mathcal{A}$ . This implies that the  $\lim_{m \rightarrow \infty} |\varphi(x_m)| = 0$ .  $\square$

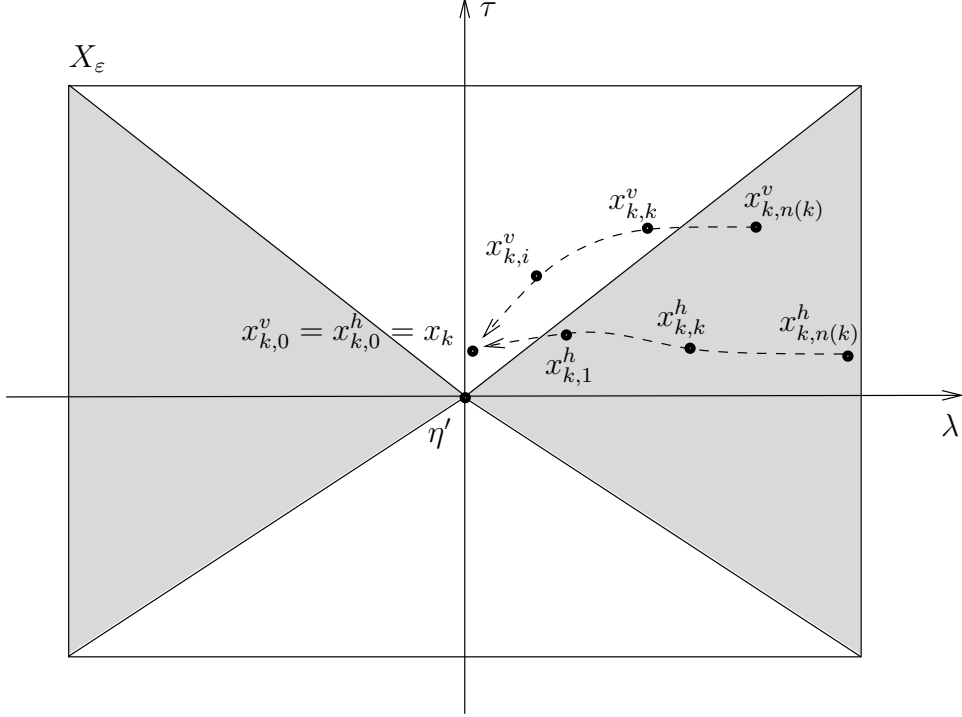
#### 4.6. Proof of Proposition 8.

**Proposition 21.** *For any  $\varepsilon > 0$  sufficiently small, there is a sequence  $\{x_k\}$  converging to  $\eta'$  such that for each  $k$ ,  $x_k$  has a preorbit of length  $k$  contained in  $C^v \cap X_\varepsilon$  and a preorbit of length  $k$  contained in  $C^h \cap X_\varepsilon$ . Moreover, any preimage of  $x_k$  that is in  $X_\varepsilon$  is in  $\Omega$ .*

*Proof.* By Lemma 19, there exists  $\varepsilon > 0$  sufficiently small so that  $X_\varepsilon \cap C^h \subset \Omega$ . For each  $k \in \mathbb{Z}_+$ , we do the following. Using Lemma 18, there exists  $\gamma \in \mathbb{Z}_+$  and  $\sigma > 0$  such that  $x_k \in X_\sigma \setminus B_\gamma$  has a preorbit  $x_{k,i}^v \subset C^v$  of length at least  $k$ . Supposing that  $\sigma$  is smaller if necessary, we can assure that  $\mathcal{R}^{-k}\{x_k\} \subset X_\varepsilon$ . Requiring that  $\gamma \geq 3$ , Lemma 17 implies that  $x_k$  has a first preimage,  $x_{k,1}^h$ , in  $C^h$ . Since  $C^h$  is backward invariant by Corollary 13,  $x_k$  has a preorbit  $x_{k,i}^h \subset C^h$  of length at least  $k$ .

It remains to show that any preimage of  $x_k$  that is in  $X_\varepsilon$  is in  $\Omega$ . First note that by Lemma 19, we can choose  $\sigma$  smaller if necessary so that  $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$ . By Lemma 15, there is an  $m \in \mathbb{Z}_+$  such that  $\mathcal{R}^{-m}(B_{\gamma+1}) \cap (X_\varepsilon \setminus X_\sigma) \subset C^h$ . Let  $0 < \tilde{\sigma} < \sigma$  be sufficiently small that if  $x \in X_{\tilde{\sigma}}$ , then  $\mathcal{R}^{-m}\{x\} \subset X_\sigma$ . Let  $x_k \in (B_{\gamma+1} \setminus B_\gamma) \cap X_{\tilde{\sigma}}$ . Using that  $B_{\gamma+1}$  is backward invariant, any preimage of  $x_k$  that is in  $X_\sigma$  will be in  $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$ . Meanwhile, by the choice of  $\tilde{\sigma}$ , any preimage that is in  $X_\varepsilon \setminus X_\sigma$  will be in  $X_\varepsilon \cap C^h \subset \Omega$ .  $\square$

*Proof of Proposition 8.* Let  $\{x_k\} \subset \Omega$  be a sequence as described in Proposition 21, and for each  $k$ , let  $\{x_{k,i}^v\}_{i=1}^k \subset C^v$  and  $\{x_{k,i}^h\}_{i=1}^k \subset C^h$  be preorbits of length  $k$  such that  $x_{k,0}^v = x_{k,0}^h = x_k$ . Each

FIGURE 4. The preorbits  $\{x_{k,i}^v\}$  and  $\{x_{k,i}^h\}$ 

preorbit  $\{x_{k,i}^h\}_{i=1}^k$  can be extended to a preorbit  $\{x_{k,i}^h\}_{i=1}^{n(k)}$  with the element  $x_{k,n(k)}^h$  being the last preimage remaining in  $X_\varepsilon$ . See Figure 4. Note that by Proposition 21 for any  $0 \leq i \leq n(k)$ , we have both  $x_{k,i}^v, x_{k,i}^h \in \Omega$ .

We first show there is a subsequence of  $\{x_{k,n(k)}^h\}$ , that converges to a point in  $\mathcal{L}_0 \setminus \{\eta', \eta\}$ . By construction,  $x_{k,n(k)}^h$  is a preimage of  $x_{k,1}^h \in C^h$ , so

$$(39) \quad x_{k,n(k)}^h \in \bigcap_{i=0}^{n(k)-1} \mathcal{R}^{-i}(C^h) \cap X_\varepsilon.$$

Also by construction,  $x_{k,n(k)}^h \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$ , which has compact closure. Thus, there is some subsequence such that  $x_{k_j,n(k_j)}^h \rightarrow x_*$  with

$$(40) \quad x_* \in \bigcap_{i=0}^{\infty} \mathcal{R}^{-i}(B_{\gamma_k}) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon.$$

However, since each  $x_{k,n(k)}^h \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$ , we must have  $|x_*| \geq \varepsilon^4$ .

By the vertical and horizontal distortion estimates in Proposition 16, preimages of  $x_k$  are escaping  $X_\varepsilon$  faster along  $x_{k,i}^h$  than  $x_{k,i}^v$ , so we also have  $x_{k,n(k)}^v \subset X_\varepsilon$ . Note that  $x_{k,i}^v$  may be in  $C^h$  for  $k \leq i \leq n(k)$ . Then using both vertical and horizontal distortion, there is a constant  $A$  so that

$$(41) \quad \text{dist}(x_{k,n(k)}^v, \eta') \leq A \text{dist}(x_k, \eta')^{\frac{1}{2^k 4^{n-k}}} \asymp A \text{dist}(x_{k,n(k)}^h, \eta')^{\frac{4^n}{2^k 4^{n-k}}} \leq A \varepsilon^{2^k},$$

which converges to 0 as  $k \rightarrow \infty$ . Thus, the sequence  $x_{k,n(k)}^v$  converges to  $\eta'$ .

By Proposition 20,  $\varphi(x_{k,n(k)}^v) \rightarrow 0$  as  $k \rightarrow \infty$ . We also have that  $|\varphi(x_{k_j,n(k_j)}^h)| \rightarrow |\varphi(x_*)| \geq \varepsilon^4$  as  $k \rightarrow \infty$ . However,  $x_{k_j,n(k_j)}^h$  and  $x_{k_j,n(k_j)}^v$  are both  $n$ th preimages of  $x_{k_j}$ , and using the invariance  $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$ , this implies  $|\varphi(x_{k_j,n(k)}^v)| = |\varphi(x_{k_j,n(k)}^h)|$  for every  $n(k)$ . Then  $0 = |\varphi(x_*)| \geq \varepsilon^4$ , a contradiction.  $\square$

#### 4.7. Proof of Theorem 4.1.

**Lemma 22.** *If  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic at  $x \in \mathcal{B} \setminus \{(\pm i, 0)\}$ , then  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic at  $\mathcal{R}(x)$ .*

*Proof.* Images of real analytic hypersurfaces under holomorphic maps were considered by Baouendi and Rothschild [1]. Suppose that  $M$  is a germ of a real analytic hypersurface in  $\mathbb{C}^N$  and  $H$  is the germ of a holomorphic map from  $\mathbb{C}^N$  to  $\mathbb{C}^N$  with  $H(0) = 0$ . The germ  $H$  is called *finite* if every point in some neighborhood of 0 has finitely many preimages. It is shown in [1, Theorem 4] that if  $H$  is finite and  $M' := H(M)$  is smooth in some neighborhood of 0, then  $M'$  is actually real analytic.

We are in the position to apply this result, since  $\mathcal{R}$  sends  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  from the neighborhood of any  $x \in \mathcal{B}$  to  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  within a smaller neighborhood of  $\mathcal{R}(x)$ . However, we must avoid the vertical lines  $z = \pm i$ , which are collapsed by  $\mathcal{R}$  to the fixed point  $(1, 0) \in \mathcal{B}$ . Away from these lines,  $\mathcal{R}$  is finite.  $\square$

*Proof of Theorem 4.1.* By Proposition 8, there is some point  $x \in \mathcal{B}$  at which  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is not real analytic. We will now use the fact that  $\mathcal{R}$  is expanding on  $\mathcal{B}$  to show that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is not real analytic in the neighborhood of any point of  $\mathcal{B}$ .

Since  $\mathcal{R}|_{\mathcal{B}}$  is  $z \mapsto z^4$ , it is expanding on  $\mathcal{B}$ , so there is some iterate  $n$  such that  $\mathcal{R}^n(U \cap \mathcal{B}) = \mathcal{B}$ . Because we assumed  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic at every point of  $U \cap \mathcal{B}$ , we can use Lemma 22 iteratively to see that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic at every point of  $\mathcal{B}$ , except perhaps at the iterated images of  $(\pm i, 0)$ . However, these consist of just the fixed point  $(1, 0)$ . To see that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic at  $(1, 0)$  note that  $(1, 0)$  is also the image of  $(-1, 0)$  under  $\mathcal{R}$ , where  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic. Thus,  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  must be real analytic at every point of  $\mathcal{B}$ , which is impossible by Proposition 8.

We now know that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is not real analytic in the neighborhood of any point of  $\mathcal{B}$ . However, it could still be real analytic in the neighborhood of some other point. We now show that this is also impossible.

Each stable manifold  $\mathcal{W}_{\text{loc}}^s(x_0)$  can be expressed as the graph of a convergent power series:

$$(42) \quad z = h(t, z_0) = \sum_{j=0}^{\infty} a_j(z_0)t^j \quad \text{where} \quad x_0 = (z_0, 0).$$

Since each  $\mathcal{W}_{\text{loc}}^s(x_0)$  depends continuously on  $z_0 \in \mathcal{B}$ , the coefficients  $a_j(z_0)$  are continuous functions of  $z_0$ . Therefore, there is a uniform radius of convergence  $\delta > 0$ . For the remainder of the proof, we suppose that the neighborhood in which  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is defined is contained in  $|t| < \delta/3$ .

Suppose  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic in a neighborhood of some  $x_1$ . Then one can express leaves of the stable foliation near  $x_1$  as graphs of some convergent power series

$$(43) \quad z = k(t, z_1) = \sum_{j=0}^{\infty} b_j(z_1)(t - t_1)^j.$$

The function  $(z_1, t) \mapsto (z, t)$ , with  $z$  given by (43), gives a parametrization of  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  near  $x_1$  with  $z_1$  varying over the real analytic arc  $\mathcal{W}_{\text{loc}}^s(\mathcal{B}) \cap \{t = t_1\}$  and  $t$  varying over some complex disc centered at  $t_0$ . Since we have assumed  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic near  $x_1$ , the parameterization is an analytic function. In particular,  $\frac{\partial^j}{\partial t^j} z$  is real analytic for each  $j \geq 0$ . Restricting to  $t = t_1$  we see that each of the coefficients  $b_j(z_1)$  is a real analytic function of  $z_1$ .

We now use this to show that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is also real analytic in a neighborhood of the unique point  $x_0$  for which  $x_1 \in \mathcal{W}_{\text{loc}}^s(x_0)$ . Since  $\mathcal{W}_{\text{loc}}^s(x_0)$  is the graph of a holomorphic function over

$|t| < \delta$ ,  $|t_1| < \delta/3$  implies that (43) converges on the disc  $|t - t_0| < \delta/2$ . In particular, each of the holomorphic discs defined by (43) crosses all the way through  $\mathcal{B}$ . As they depend real analytically on  $z_1$ , this implies that  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  is real analytic in a neighborhood of  $x_0 \in \mathcal{B}$ , which is not possible.  $\square$

## 5. PHYSICAL INTERPRETATION

In this section we will relate Theorems 2' to the Ising Model on the DHL. We refer the reader to [5, 4] for physical background. The DHL is a sequence of graphs  $\Gamma_n$  obtained in a self-similar way. Associated to each graph is a partition function  $Z_n(z, t)$  whose zeros

$$\mathcal{S}_n^c := \{(z, t) \in \mathbb{C}^2 : Z_n(z, t) = 0\}$$

describe the singularities of the Ising model associated to  $\Gamma_n$ . They are called the *Lee-Yang-Fisher zeros*. The actual physics is described by the limit  $n \rightarrow \infty$ . It is proved in [4] that the limiting distribution of zeros exists as a closed, positive  $(1, 1)$ -current  $\mathcal{S}^c$  on  $\mathbb{P}^2$ . In fact,  $\mathcal{S}^c = \frac{1}{2}\Psi^*S$ , where  $S$  is the Green current for  $R$ . The support of  $\mathcal{S}^c$  describes locus where phase transitions occur in  $\mathbb{C}^2$ .

It is shown in [4] that at low complex temperatures  $\text{supp } \mathcal{S}^c$  coincides with  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$ . Combining Theorem 2' with the work from [4] gives the following:

**Corollary 23.** *At low complex temperatures ( $|t|$  small), the locus of phase transitions for the Ising model on the DHL is a 3 real-dimensional manifold that is  $C^\infty$  but not real analytic.*

A preferred subset of the Lee-Yang-Fisher zeros is obtained by requiring that  $t \in [0, 1]$ , which correspond to “physical” temperatures. The Lee-Yang Circle Theorem [25, 15] asserts that for each  $n$  and fixed  $t_0 \in [0, 1]$ , zeros of partition function  $Z_n(z, t_0)$  corresponding to  $\Gamma_n$  lie on the unit circle  $\mathbb{T}_{t_0} := \{|z| = 1, t = t_0\}$ . Let

$$\mathcal{C} = \{|z| = 1, t \in [0, 1]\}.$$

The *Lee-Yang zeros* are defined by

$$\mathcal{S}_n := \{(z, t) \in \mathcal{C} : Z_n(z, t) = 0\}.$$

Isakov [12] proved for any  $t_0 > 0$  sufficiently small the free energy for the Ising model on the  $\mathbb{Z}^d$  lattice with  $d > 1$  does not have analytic continuation through any point of the circle  $\mathbb{T}_{t_0}$ . This implies that the limiting distribution of Lee-Yang zeros for the  $\mathbb{Z}^d$  lattice with  $d > 1$  does not have real analytic density in the neighborhood of any point of the circle  $t = t_0$ . In the remainder of this section, we discuss how Corollary 23 can be related to Isakov’s result.

One can check that  $\mathcal{R}$  maps the Lee-Yang cylinder  $\mathcal{C}$  into itself, with the Lee-Yang zeros corresponding to  $\Gamma_{n+1}$  obtained by pulling back the Lee-Yang zeros corresponding to  $\Gamma_n$  under  $\mathcal{R}|_{\mathcal{C}}$ . The map  $\mathcal{R}: \mathcal{C} \rightarrow \mathcal{C}$  was also studied previously by Bleher and Žalys [3].

In [5], Bleher, Lyubich, and Roeder describe the limiting distribution of Lee-Yang zeros for the DHL; let us provide a very brief summary. Let  $\mathcal{C}_1 := \mathcal{C} \setminus \{t = 1\}$ . It was shown that  $\mathcal{R}: \mathcal{C}_1 \rightarrow \mathcal{C}_1$  is partially hyperbolic, with a unique central foliation  $\mathcal{F}^c$  which is vertical (with respect to a suitable cone field) on  $\mathcal{C}_1$ . In particular, one can define the  $\mathcal{F}^c$  holonomy map  $g_t: \mathbb{T}_0 \rightarrow \mathbb{T}_t$ . The limiting distribution of Lee-Yang zeros at temperature  $t_0 \in [0, 1)$  is obtained as the pushforward  $\mu_t = g_{t_0*}\text{Leb}$ , where  $\text{Leb}$  is the normalized Lebesgue measure on  $\mathbb{T}_0$ .

In a neighborhood of  $\mathcal{B}$ ,  $\mathcal{F}^c$  coincides with the stable foliation of  $\mathcal{B}$ , which is a union of the real analytic curves  $\mathcal{W}_{\text{loc}}^s(x) \cap \mathcal{C}$ , taken over  $x \in \mathcal{B}$ . It is shown in [4, Lemma 3.2] that the stable foliation of  $\mathcal{B}$  within  $\mathcal{C}$  has the same regularity that the stable manifold  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  does as a submanifold of  $\mathbb{C}^2$ . (In fact,  $\mathcal{W}_{\text{loc}}^s(\mathcal{B})$  was shown to be a  $C^\infty$  manifold in [4] by first showing that the stable foliation of  $\mathcal{B}$  within  $\mathcal{C}$  is  $C^\infty$ .)

Therefore, Theorem 2' implies that the central foliation is not real analytic at low temperatures. Moreover, by [5], an open dense set of points from  $\mathcal{C}$  have orbits converging to  $\mathcal{B}$ . Since  $\mathcal{F}^c$  is invariant, this implies the following:

**Theorem 24.**  $\mathcal{F}^c$  is not real analytic in the neighborhood of any point of  $\mathcal{C}$ .

Using the holonomy description of the limiting distribution of Lee-Yang zeros, we find the following modest analog of Isakov's Theorem for the DHL:

**Corollary 25.** For any  $z = e^{i\phi} \in \mathcal{B}$ , there is a dense set of  $t_0 \in [0, 1]$  so that the limiting distribution of Lee-Yang zeros within  $\mathbb{T}_{t_0}$  does not have real analytic density at  $(t_0, \phi)$ .

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