PART 6: CONSTRUCTION OF PARTICLE SYSTEMS IN INFINITE VOLUME

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We construct, using the method of Liggett and Spitzer [3] and Andjel [1], zero-range and exclusion particle systems on $\mathbb{Z}^d$. The approach is to take a limit of the processes restricted to large but finite sets. Other consruction techniques are mentioned in the Notes section.

1. What does it mean to construct a process?

In Part 1, on finite or countable state spaces Ω, 'construction of the process' meant construction as a Markov chain, specifying transition probabilities $P_{x,y}(t) = P(\eta(t) = y|\eta(0) = x)$, from which probabilities $P(\eta(t) \in A|\eta(0) = x)$ could be found for $A \subset \Omega$. One then could write down the semigroup operator P_t, acting on bounded functions, and understand its connection to a generator L in terms of Backward and Forward equations.

On more exotic spaces Ω, to construct a Markov process usually means building a semigroup operator P_t, that is an operator with the 'Chapman-Kolmogorov' property $P_{t+s} = P_t P_s$, which acts on a collection of function on Ω, and relating P_t to a generator L through backward and forward evolution equations. In this context, one can usually associate, by Kolmogorov's extension theorem, a probability $P^\eta[\eta(t) \in d\zeta]$ on the Borel sets in Ω. Sometimes, we will want the semigroup P_t to have certain properties, such as 'strong continuity' or the 'Feller property' (which we will explore in the next part), but this is not guaranteed.

For the particle systems we have studied, exclusion and zero-range models, we would like to extend the background space $\mathbb{T}_N^d$ to $\mathbb{Z}^d$. Then, the configuration space would be $\Omega = \{0,1\}^{\mathbb{Z}^d}$ for exclusion systems, and $\Omega = \{0,1,2,\ldots\}^{\mathbb{Z}^d}$ for zero-range models. With respect to exclusion systems, Ω is compact, which is helpful and allows different ways to construct the process. However, for zero-range models, since Ω now is not compact, some care must be taken and assumptions on the parameters, namely the rate function g and transition probability p, should be made.

Interestingly, if g is in general unbounded, but Lipschitz, not all configurations $\eta \in \Omega$ are 'allowed'. In other words, we will construct the zero-range process on a strict subset $\Omega' \subset \Omega$, so that once begun in Ω', the process will stay in Ω'. Part of the reason for this restriction lies in that if the rate function is large, particles jump faster, and the process may 'blow up' if started from an unsuitable configuration.

On the other hand, if g is bounded, then the process can be constructed on the full Ω by a different method [4].

2. Zero-range model and statement of results

We will assume in the following that $g : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ satisfies $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$, and that g is Lipschitz: There is a constant a_0 such that

(LIP) $\sup_{k\geq 0} |g(k+1) - g(k)| \ \leq \ a_0$.

Also, we will take the jump probability p on $\mathbb{Z}^d$ to be translation invariant $p(x, y) = p(0, y - x) = p(y - x)$, and finite-range: There is $R < \infty$ such that $p(x) = 0$ for $|x| > R$.

We now specify the allowed configuration space Ω'. For $x \in \mathbb{Z}^d$, let

$$\beta(x) \ = \ \sum_{n\geq 0} \frac{p^{(n)}(x, 0)}{2^n}$$

where $p^{(n)}(x, y)$ is the probability of reaching $y \in \mathbb{Z}^d$ in n steps from x. Observe that

$$\sum_{y\in\mathbb{Z}^d} p(x, y)\beta(y) \ \leq \ 2\beta(x). \tag{2.1}$$

Exercise 2.1. Show (2.1).

Define for $\eta, \zeta \in \Omega = \{0, 1, 2, \ldots\}^{\mathbb{Z}^d}$ the norm

$$\|\eta - \zeta\| \ = \ \sum_{x\in\mathbb{Z}^d} |\eta(x) - \zeta(x)|\beta(x).$$

The collection of allowed configurations will be

$$\Omega' \ = \ \big\{\eta \in \Omega \big| \|\eta\| = \sum_{x\in\mathbb{Z}^d} \eta(x)\beta(x) < \infty\big\}.$$

We now define a class of 'Lipschitz' functions which will be the function on which we construct the process. We say that $f : \Omega' \to \mathbb{R}$ is Lipschitz if thre is a constant c such that

$$|f(\eta) - f(\zeta)| \ \leq \ c\|\eta - \zeta\|$$

for all $\eta, \zeta \in \Omega'$. Let $c(f)$ be the smallest such constant c.

Then, denote by $\mathcal{L}$ the collection of all Lipschitz functions on Ω'. Although $\mathcal{L}$ is not a Banach space, it will suffice for what follows, and will allow an extension of the process to L^2 with respect to an invariant measure, which is a Banach space, in the next Part.

Define operator L, which we will identify later as our generator, on functions in $\mathcal{L}$ by

$$(Lf)(\eta) \ = \ \sum_{x,y\in\mathbb{Z}^d} p(y)g(\eta(x))\big[f(\eta^{x,x+y}) - f(\eta(x))\big]$$

where $\eta \in \Omega'$.

Lemma 2.2. *The operator L is well defined for $f \in \mathcal{L}$ and $\eta \in \Omega'$, and*

$$|Lf(\eta)| \ \leq \ 3a_0c(f)\|\eta\|.$$

Proof. The estimate follows from

$$\begin{aligned} |Lf(\eta)| &\le \sum_{x,y} p(y) g(\eta(x)) \big| f(\eta^{x,x+y}) - f(\eta) \big| \\ &\le c(f) \sum_{x,y} p(y) g(\eta(x)) \big| \beta(x+y) - \beta(x) \big| \\ &\le a_0 c(f) \sum_{x,y} p(y) \eta(x) \big(\beta(x+y) + \beta(x) \big) \\ &\le 3a_0 c(f) \|\eta\| \end{aligned}$$

using (2.1) in the last step. □

Remark 2.3. When there are only a finite number of particles in the system, P_t and L are the semigroup and generator of the associated countable state Markov chain $\eta(t)$.

We now come to the main theorems.

Theorem 2.4. *There exists a semigroup P_t on $\mathcal{L}$ such that $P_t f(\eta) = E^\eta[f(\eta(t))]$ for $f \in \mathcal{L}$ and η which corresponds to a finite number of particles. This semigroup satisfies*

$$\big| P_t f(\eta) - P_t f(\zeta) \big| \le c(f) e^{4a_0 t} \|\eta - \zeta\|$$

for $\eta, \zeta \in \Omega'$ and $f \in \mathcal{L}$, and also

$$P_t f(\eta) = f(\eta) + \int_0^t L P_s f(\eta) ds$$

for $\eta \in \Omega'$ and $f \in \mathcal{L}$.

More properties are given in the following result which will be helpful in later Parts.

Theorem 2.5. *The semigroup P_t is such that for $f \in \mathcal{L}$ and $\eta \in \Omega'$ we have*

(i) $|P_t f(\eta) - f(\eta)| \le (4a_0)^{-1} c(f) \|\eta\| (e^{4a_0 t} - 1)$
(ii) $\lim_{t \downarrow 0} t^{-1} \big[P_t f(\eta) - f(\eta) \big] = L f(\eta)$
(iii) $L P_t f(\eta) = P_t L f(\eta)$.

2.1. **The meaning of Theorem 2.4.** One may check that cylinder functions, $f(\eta) = \prod_{j=1}^k 1_{A_j}(\eta(x_j))$ for $A_j \subset \{0, 1, 2, \ldots\}$ and $x_j \in \mathbb{Z}^d$, are Lipschitz functions. Since, for a given $\eta \in \Omega'$, one may approximate η by finite configurations ζ^n such that $\|\eta - \zeta^n\| \downarrow 0$ as $n \uparrow \infty$, by Theorem 2.4, the semigroup $P_t f(\eta)$ may be computed: $P_t f(\eta) = \lim E^{\zeta^n}[f(\eta_t)]$. Therefore, the finite dimensional distributions of $(\eta_t(x), t \ge 0, x \in \mathbb{Z}^d)$ for a given initial configuration η may be identified.

Then, by Kolmogorov's extension theorem, we have a probability measure $P^\eta[\eta(t) \in d\zeta]$ on Borel sets Ω (not necessarily for the moment Ω'!) for each $t \ge 0$ and $\eta \in \Omega'$ with the given marginal distributions,

$$P_t f(\eta) = \int P^\eta[\eta_t \in d\zeta] f(\zeta) = E^\eta[f(\eta_t)] \qquad (2.2)$$

for $f \in \mathcal{L}$ which is a local function, that is f depends only on a finite number of coordinates. Such functions are approximated by linear combinations of cylinder functions.

Lemma 2.6. *The measure $P^\eta[\eta_t \in d\zeta]$ concentrates on Ω' when $\eta \in \Omega'$.*

Proof. We apply Theorem 2.4 to the function $f(\eta) = \|\eta\|$ which is clearly Lipschitz with $c(f) = 1$. Hence,

$$e^{4a_0 t}\|\eta\| \;\le\; |P_t f(\eta)| \;=\; |E^\eta[f(\eta_t)]| \;\;=\; E^\eta[\|\eta_t\|]$$

and so $\|\eta_t\| < \infty$ a.s. starting from η. □

Lemma 2.7. *For $\eta \in \Omega'$, we may identify for $f \ge 0$ or $f \in \mathcal{L}$ that*

$$P_t f(\eta) \;=\; \int P^\eta[\eta_t \in d\zeta] f(\zeta) \;=\; E^\eta[f(\eta_t)].$$

Proof. When $f \ge 0$, one can approximate f by simple functions which are in particular local functions in $\mathcal{L}$. Then, by monotone convergence, one could take a limit in (2.2), and define $P_t f(\eta)$ in this way.

When $f \in \mathcal{L}$, however, not necessarily positive, we may still approximate it by local functions in $\mathcal{L}$. Since $|f(\zeta)| \le c(f)\|\zeta\| + |f(\zeta^0)|$ when $\zeta \in \Omega'$, the point is that, by Lemma 2.6, we can pass to the limit in (2.2) to define $P_t f(\eta)$. □

3. Construction estimates on a finite cube

The strategy to construct a semigroup P_t on the infinite volume is first to obtain estimates on the semigroup and generator which are well defined when the space is finite, and then use the estimates to define P_t as the volume grows. Throughout this section, we will assume that

$$A \;=\; \{x : |x_i| \le L, 1 \le i \le d\}$$

is a box of width $2L + 1$. Let $P_t = P_t^{(A)}$ and $\eta_t = \eta_t^{(A)}$ be the zero-range process corresponding to transition probablity $p(x, y)$ on A.

Lemma 3.1. *For $t \ge 0$, and $y \in A$, we have*

$$E^\eta[\eta_t(y)] \;\le\; \sum_{x\in A} \eta(x) \sum_{\ell=0}^{\infty} \frac{(a_0 t)^\ell}{\ell!} p^{(\ell)}(x, y)\Big].$$

Here, $p^{(\ell)}(x, y)$ is the ℓ-step transition probability from x to y.

Proof. We will first couple the zero-range process to a continuous-time multitype branching process $\eta^*(t)$ on $\{0, 1, 2, \ldots\}^A$ with generator

$$L^* f(\eta) \;=\; \sum_{x, x+y\in A} a_0 \eta(x) p(x, x + y)\big[f(\eta + \delta_{x+y}) - f(\eta)\big]$$

where $\eta + \delta_z$ is the configuration which adds coordinatewise to η a particle to site z. An inspection of the formula reveals that η^* is such that each particle gives birth to a new particle at rate a_0. This new particle is then displaced by y with probability $p(y)$. Alternatively, each particle gives birth to two new particles before it dies; one is kept at the birth location, and the other is displaced by y.

One can couple $\eta(t)$ and $\eta^*(t)$ as follows: Whenever a zero-range particle displaces from x to $x + y$, arrange that the branching process gives birth at x and creates a new particle at $x + y$. In this way, if the two processes are started from the same initial configuration, then $\eta(y) \le \eta^*(y)$ for all $y \in A$ and $t \ge 0$.

We now calculate $E^\eta[\eta_t^*(y)]$. One may associate to a particle initially at x a binary progeny tree. In the first generation, there are two branches corresponding to a particle renewed at x and another placed at z with probability $p(z - x)$. In the second generation and so on, the particle at z splits into two particles, one kept at z and the other displaced according to p; the particle at x splits like in the first generation. Since the split times are exponential(a_0), the total number of generations is a Poisson($a_0 t$) variable. From the form of the tree, in the rth generation, in mean-value, there are $\sum_{\ell=0}^{r} p^{(\ell)}(x,y)\binom{r}{\ell}$ particles which are at location y. Now, there are $\eta(x)$ particles at x initially. Hence, adding over all particles in A, we have

$$\begin{aligned}
E^\eta\big[\eta_t^*(y)\big] &\leq \sum_{x\in A}\eta(x)\sum_{r\geq 0} e^{-a_0 t}\frac{(a_0 t)^r}{r!}\sum_{\ell=0}^{r} p^{(\ell)}(x,y)\binom{r}{\ell} \\
&= \sum_{x\in A}\eta(x)\sum_{\ell=0}^{\infty}\sum_{r\geq \ell} e^{-a_0 t}\frac{(a_0 t)^r}{r!}\binom{r}{\ell} p^{(\ell)}(x,y) \\
&= \sum_{x\in A}\eta(x)\sum_{\ell=0}^{\infty}\frac{(a_0 t)^\ell}{\ell!} p^{(\ell)}(x,y).
\end{aligned}$$

The last line follows as the factorial moment of order ℓ for a Poisson variable X with parameter λ is $E[X(X-1)\cdots(X-\ell+1)] = \lambda^\ell$.

The proof now follows as $E^\eta[\eta_t(y)] \leq E^\eta[\eta^*(y)]$. □

Lemma 3.2. *For $t \geq 0$ and $f \in \mathcal{L}$, $P_t f \in \mathcal{L}$ and $c(P_t f) \leq c(f)e^{3a_0 t}$.*

Proof. Consider the basic coupling given in Part 4 which is the joint process on $[\{0,1,2,\ldots\}^A]^2$ generated by

$$\begin{aligned}
\bar{L}\phi(\eta,\zeta) &= \sum_{x,x+y\in A}\min\{g(\eta(x)),g(\zeta(x))\}p(x,x+y)\big[\phi(\eta^{x,x+y},\zeta^{x,x+y})-\phi(\eta,\zeta)\big] \\
&\quad + \sum_{x,x+y\in A}\big(g(\eta(x))-g(\zeta(x))\big)^+ p(x,x+y)\big[\phi(\eta^{x,x+y},\zeta)-\phi(\eta,\zeta)\big] \\
&\quad + \sum_{x,x+y\in A}\big(g(\zeta(x))-g(\eta(x))\big)^+ p(x,x+y)\big[\phi(\eta,\zeta^{x,x+y})-\phi(\eta,\zeta)\big].
\end{aligned}$$

This defines a Markov chain on a countable state space; let $\bar{P}_t$ be the coupled process semigroup. Both marginals are zero-range process, and if initially $\eta \leq \zeta$, then the coordinatewise ordering is preserved, $\eta(t) \leq \zeta(t)$.

Now write

$$\begin{aligned}
|P_t f(\eta) - P_t(\zeta)| &= |\bar{P}_t(f(\eta)-f(\zeta)| \\
&\leq c(f)\bar{P}_t\|\eta-\zeta\|.
\end{aligned}$$

We will like to give an estimate of the right-side in terms of its derivative. Note

$$\|\eta^{x,x+y}-\zeta\| \;=\; \|\eta-\zeta^{x,x+y}\| \;=\; |\beta(x+y)-\beta(x)\|.$$

Then, adding together the positive and negative parts of $g(\eta(x)) - g(\zeta(x))$, we obtain

$$\begin{aligned}
\bar{L}\|\eta-\zeta\| &\leq a_0\sum_{x,x+y\in A}|\eta(x)-\zeta(x)|p(x,x+y)|\beta(x+y)+\beta(x)| \\
&\leq 3a_0\|\eta-\zeta\|
\end{aligned}$$

noting that $\sum_y p(x,x+y)\beta(x+y) \leq 2\beta(x)$.

Hence, since $(d/dt)\bar{P}_t\|\eta-\zeta\| = \bar{P}_t\bar{L}\|\eta-\zeta\|$, by comparison or Gronwall's lemma, we have

$$|P_t f(\eta) - P_t f(\zeta)| \;\leq\; c(f)e^{3a_0 t}\|\eta-\zeta\|$$

which finishes the proof. □

Let now (P_t^1, L_1) and (P_t^2, L_2) be zero-range processes on A according to jump probabilities p_1 and p_2 respectively on A which satisfy (2.1).

Lemma 3.3. *For $f \in \mathcal{L}$, we have*

$$|(L_1 - L_2)f(\eta)| \ \le \ a_0 c(f) \sum_{x, x+y \in A} \eta(x)|p_1(x, x+y) - p_2(x, x+y)||\beta(x) + \beta(x+y)|.$$

Lemma 3.4. *We have*

$$P_t^1 f(\eta) - P_t^2 f(\eta) \ = \ \int_0^t P_s^1 \big[L_1 - L_2\big] P_{t-s}^2 f(\eta) ds.$$

Exercise 3.5. Prove Lemmas 3.3 and 3.4.

Lemma 3.6. *For $f \in \mathcal{L}$, we have*

$$\begin{aligned} &|P_t^1 f(\eta) - P_t^2 f(\eta)| \\ &\le \ a_0 c(f) \int_0^t e^{3a_0(t-s)} \sum_{x \neq x+y} P_s^1 \eta(x) \\ &\qquad\qquad |p_1(x, x+y) - p_2(x, x+y)||\beta(x) + \beta(x+y)| ds \\ &\le \ a_0 c(f) \int_0^t e^{3a_0(t-s)} \sum_{x \neq x+y} \Big[\sum_{z \in A} \eta(z) \sum_{\ell \ge 0} \frac{(a_0 s)^\ell}{\ell!} p_1^{(\ell)}(z, x)\Big] \\ &\qquad\qquad |p_1(x, x+y) - p_2(x, x+y)||\beta(x) + \beta(x+y)| ds. \end{aligned} \tag{3.1}$$

Proof. The inequalities follow by applying Lemma 3.4, Lemma 3.3, and then Lemma 3.1. □

4. Extension to $\mathbb{Z}^d$

Let $A = A_L$ increase to $\cup_{L \ge 1} A_L = \mathbb{Z}^d$. For transition probability p on $\mathbb{Z}^d$, define the truncations

$$q_L(x, z) \ = \ \begin{cases} p(x, z) & \text{if } x, z \in A_L, x \neq z \\ 1 & \text{if } x = y \notin A_L \\ p(x, x) + \sum_{w \notin A_L} p(x, w) & \text{if } x = y \in A_L. \end{cases}$$

Note that q_L does not allow transitions from A_L to A_L^c. It will be useful to note that $q_L(x, z) \le p(x, z) + \delta_{x,z}$. Hence, q_L satisfies (2.1) with constant 3 instead of 2.

Let P_t^L be the semigroup corresponding to q_L on A.

Proposition 4.1. *For $f \in \mathcal{L}$ and $\eta \in \Omega'$, we have $\lim_{L \uparrow \infty} P_t^L f(\eta)$ converges uniformly on bounded t sets, compact η sets, and sets of functions with bounded $c(f)$.*

Proof. We show that $P_t^L f(\eta)$ forms a Cauchy sequence with the uniformity properties.

With $p_1 = q_L$ and $p_2 = q_{L'}$, where $L \le L'$, we have that the integrand in (3.1) is bounded by

$$\begin{aligned} & e^{3a_0(t-s)} \sum_{x\in A_{L'}} \Big[\sum_{z\in A_{L'}} \eta(z) \sum_{\ell\ge 0} \frac{(a_0 s)^\ell}{\ell!} q_L^{(\ell)}(z,x) \Big] (2\cdot 3)\beta(x) \\ \le\ & 6e^{3a_0(t-s)} \sum_{z\in A_{L'}} \eta(z) \sum_{\ell\ge 0} \frac{(2a_0 s)^\ell}{\ell!} \beta(z) \\ \le\ & 6e^{3a_0 t}\|\eta\|. \end{aligned}$$

Here, we used (2.1) several times. Hence, the integrand is dominated.

To show pointwise convergence for each $0 \le s \le t$, note first, for $x \ne x+y$, that

$$|q_L(x,x+y) - q_{L'}(x,x+y)| \ = \ -\big(q_L(x,x+y) - q_{L'}(x,x+y)\big)$$

since in a sense $q_{L'}$ covers q_L off the diagonal $x = x+y$. Then, the integrand is written

$$e^{3a_0(t-s)} \sum_{x\ne x+y} P_s^L \eta(x) \big[q_{L'}(x,x+y) - q_L(x,x+y)\big]\big(\beta(x) + \beta(x+y)\big). \tag{4.1}$$

Consider the term

$$\begin{aligned} & e^{3a_0(t-s)} \sum_{x\ne x+y} P_s^L \eta(x) q_{L'}(x,x+y)\big(\beta(x)+\beta(x+y)\big) \qquad (4.2) \\ & = \ e^{3a_0(t-s)} \sum_{x\ne x+y} \Big[\sum_{z\in A} \eta(z) \sum_{\ell\ge 0} \frac{(a_0 s)^\ell}{\ell!} p_1^{(\ell)}(z,x)\Big] q_{L'}(x,x+y)\big(\beta(x)+\beta(x+y)\big) \end{aligned}$$

and the counterpart term where $q_L(x,x+y)$ replaces $q_{L'}(x,x+y)$.

We want to take $L, L' \uparrow \infty$, and show these expressions converge to the same limit. We may majorize $q_L(x,w), q_{L'}(x,w) \le r(x,w) := p(x,w) + \delta_{x,w}$, and substituting in r into (4.2), by the argument already given in the first part of the proof, we see it is bounded.

On the other hand, both $q_L, q_{L'}$ converge to p which shows (4.2) and its counterpart both tend to the same limit by dominated convergence. Therefore, (4.1) vanishes as $L, L' \uparrow \infty$.

Plugging into (3.1), we see that the convergence is uniform as desired. ☐

We may now define, for $f \in \mathcal{L}$ and $\eta \in \Omega'$ that

$$P_t f(\eta) \ = \ \lim_{L\uparrow\infty} P_t^L f(\eta).$$

Moreover, by Lemma 3.2, since q_L satisfies (2.1) with constant 3, we have

$$c(P_t f) \ \le \ c(f) e^{4a_0 t}.$$

We now come to the proof of Theorem 2.4.

Proof of Theorem 2.4. We first establish the semigroup property for P_t: Namely, $P_t P_s = P_{t+s}$. Already this property holds for P_t^L. We need to show for $f \in \mathcal{L}$ and $\eta \in \Omega'$ that

- $\lim_{L\uparrow\infty} [P_t - P_t^L] P_s^L f(\eta) = 0$
- $\lim_{L\uparrow\infty} P_t [P_s^L - P_s] f(\eta) = 0.$

The first limit follows from the uniform convergence in Proposition 4.1 as $c(P_s^L f) \leq c(f)e^{4a_0 s}$ uniformly in L.

For the second limit, we note that for fixed t and $\eta \in \Omega'$, by construction, we may find by Kolmogorov's theorem (as discussed in subsection 2.1) a probability μ on Ω such that $\int \|\zeta\|\mu(d\zeta) < \infty$ and, for $h \in \mathcal{L}$,

$$P_t h(\eta) \;=\; \int h(\zeta)\mu(d\zeta).$$

Now observe that

$$|P_s^L f(\eta) - P_s f(\eta)| \;=\; |P_s^L(f(\eta) - f(0)) - P_s(f(\eta) - f(0))| \;\leq\; 2c(f)e^{4a_0 t}\|\eta\|.$$

Hence, by dominated convergence,

$$\int [P_s^L f(\zeta) - P_s f(\zeta)]\mu(d\zeta) \;\to\; 0.$$

To establish the integral property, note that it holds for P_t^L:

$$P_t^L f(\eta) - f(\eta) \;=\; \int_0^t L_L P_s^L f(\eta) ds.$$

We can already pass to the limit on the left-side. To pass on the right-side we need to show that

$$L_L P_s^L f(\eta) - L P_s f(\eta) \;\to\; 0 \tag{4.3}$$

and that the the integrand is dominated. Domination holds as

$$|L_L P_s^L f(\eta)| \;\leq\; 3c(f)e^{4a_0 t}\|\eta\|. \tag{4.4}$$

We leave to the reader to show (4.3). □

Exercise 4.2. Show (4.3) using estimates developed, and the explicit form of L.

5. Proof of Theorem 2.5

We will use the previous estimates to show Theorem 2.5.

Proof of (i). From (4.4), we have that

$$|LP_s f(\eta)| \;\leq\; 3c(f)e^{4a_0 t}\|\eta\|.$$

Now, plugging into the integral expression in Theorem 2.4, and integrating, we obtain (i).

Proof of (ii). If we can show that $LP_s f(\eta)$ is continuous at $t = 0$, then (ii) follows from the integral formula in Theorem 2.4. Now, from (i), P_t is continuous at $t = 0$. Write

$$LP_s f(\eta) \;=\; \sum_{x,x+y} g(\eta(x))p(x, x+y)\big[P_s f(\eta^{x,x+y}) - P_s f(\eta)\big].$$

To show continuity of $LP_s f(\eta)$, we dominate

$$|P_s f(\eta^{x,x+y}) - P_s f(\eta)| \;\leq\; c(P_s f)\|\eta^{x,x+y} - \eta\| \;\leq\; c(f)e^{3a_0 t}[\beta(x+y) + \beta(x)].$$

The right-side bound is summable:

$$\sum_{x,x+y} g(\eta(x))p(x, x+y)[\beta(x+y) + \beta(x)] \;\leq\; 3a_0\|\eta\|.$$

Hence, the desired continuity follows from dominated convergence.

Proof of (iii). Write

$$\begin{aligned} LP_t f(\eta) &= \lim_{s\downarrow 0} t^{-1}\big[P_s P_t f(\eta) - P_t f(\eta)\big] \\ &= \lim_{s\downarrow 0} P_t \frac{P_s f - f}{s}(\eta) \\ &= P_t L f(\eta). \end{aligned}$$

The first equality is from part (ii). The second equation is from the semigroup property proved in Theorem 2.4. The third equality is from dominated convergence, writing $P_t(\eta) = E^\eta[h(\eta_t)]$. To justify the convergence, we note the pointwise convergence is established already in part (ii). The domination is as follows using part (i): For $0 < s \le 1$,

$$\big|s^{-1}\big[P_s f - f\big](\eta)\big| \ \le \ (4a_0)^{-1}c(f)\|\eta\| s^{-1}|e^{4a_0 s} - 1| \le 2c(f)e^{4a_0}\|\eta\|.$$

6. Notes

The material follows [1] which is almost always cited when working with zero-range processes. On the other hand, other construction methods exist. For instance, if g is bounded, one can construct the semigroup by the Hille-Yosida theorem as in [4]. [2] gives another way, along the lines developed above, with different estimates, to construct the semigroup.

References

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