LECTURE 2: EMPIRICAL MEASURES, AND A FIRST LOOK AT HYDRODYNAMICS OF THE SIMPLE EXCLUSION PROCESSES

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We introduce a weaker notion of 'hydrodynamics' via empirical measures, which will be used with respect to more general interacting particle systems. Then, after a preliminary discussion of certain martingales in the Markov chain context, we discuss the hydrodynamics of simple exclusion processes.

1. Empirical measures and view of 'hydrodynamics' as a LLN

We now recast the derivation of 'hydrodynamics' in systems of independent random walks in terms of the space-time scaled empirical distribution

$$\pi^N_{v(N)t} \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\delta_{x/N}\eta_{v(N)t}(x).$$

Here, $\pi^N_{v(N)t}$ is a member of $\mathcal{M}_+(\mathbb{T}^d)$, the nonnegative measures on $\mathbb{T}^d$. Before, in the previous section, we derived the distribution of the particle numbers at later times, starting from a local equilibrium measure. However, it will be easier in what follows to consider the weaker notion of the asymptotic behavior of the empirical distribution. In this sense, what is meant by 'hydrodynamics' is a law of large numbers, characterizing 'first-order' behavior.

Let G be a smooth, bounded function, and write

$$\langle G,\pi^N_{v(N)t}\rangle \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x). \tag{1.1}$$

Recall that the distribution of $\eta_{v(N)t}$ starting from $\nu^N_{\rho_0(\cdot)}$ is a product of Poisson measures with intensity $\psi_{N,v(N)t}(\cdot) = E\big[\rho_0\big(N^{-1}(x-Z^N_{v(N)t})\big)\big]$, where the expectation is with respect to the random walk $Z^N_\cdot$. Then, in mean-value, we have

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x)\Big] &= \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\eta_{v(N)t}(x)]\\ &= \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)E\big[\rho_0\big(N^{-1}(x-Z_{v(N)t})\big)\big].\end{aligned}$$

Depending on whether $m\neq 0$ or $m=0$, in which case $v(N)=N$ or $v(N)=N^2$, the last quantity as before tends respectively to

$$\int \rho_0(u-mt)G(u)du \;\;\text{or}\;\; \int G(u)\int\bar\rho_0(w)G_t(u-w)dwdu. \tag{1.2}$$

However, the variance of (1.1), since under μ^N the occupation numbers $\eta_{v(N)t}$ are independent Poisson variables with intensity $\psi_{N,v(N)t}(x)$, equals

$$\begin{aligned}\mathrm{Var}_{\mu^N}\Big[\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} G(x/N)\eta_{v(N)t}(x)\Big] &= \frac{1}{N^{2d}}\sum_{x\in\mathbb{T}_N^d} G^2(x/N)\psi_{N,v(N)t}(x)\\ &= O(N^{-d}) \ \to\ 0.\end{aligned}$$

Hence, $\langle G, \pi^N_{v(N)t}\rangle$ converges to the expressions in (1.2) in probability, depending on the drift m.

In particular, we have shown, with respect to the initial distribution μ^N for the process η_0, the empirical measure $\pi^N_{v(N)t}$, a random element of $\mathcal{M}_+(\mathbb{T}^d)$, converges in probability to the deterministic measure $\rho(t,u)du$ corresponding to the macroscopic space-time mass evolution.

Here, the topology on $\mathcal{M}_+(\mathbb{T}^d)$ used is as follows: Consider $C(\mathbb{T}^d)$, the space of real continuous functions on $\mathbb{T}^d$ endowed with the sup-metric. Let $\{f_k : k \geq 1\}$ be a dense, countable family of continuous functions in $C(\mathbb{T}^d)$. Then, define the distance $\delta(\mu,\nu)$ on $\mathcal{M}_+(\mathbb{T}^d)$ by

$$\delta(\mu,\nu) \ = \ \sum_{k=1}^\infty \frac{1}{2^k}\frac{|\langle\mu,f_k\rangle - \langle\nu,f_k\rangle|}{1+|\langle\mu,f_k\rangle - \langle\nu,f_k\rangle|}.$$

Hence, capturing the limit behavior of $\langle G,\pi^N_{v(N)t}\rangle$ is enough for each continuous function G.

2. Why is it called 'hydrodynamics'?

At this point, we comment on why the scaling limit is called a 'hydrodynamic limit'. The origins go back to the study of $L = \rho N$ identical mass 1 particles with certain positions $\{q^\ell = \langle q_i^\ell : i = 1,2,3\rangle\}_{\ell=1}^L$ and momenta $\{p^\ell = \langle p_i^\ell : i = 1,2,3\rangle\}_{\alpha=1}^L$ moving on a torus $N\mathbb{T}^3$, with width N, according to Newtonian dynamics:

$$\begin{aligned}\frac{dq_i^\ell}{dt} = \frac{\partial H}{\partial p_i^\ell} &= p_i^\ell \ \text{ and}\\ \frac{dp_i^\ell}{dt} = -\frac{\partial H}{\partial q_i^\ell} &= -\sum_k V_i(q^\ell - q^k),\end{aligned}$$

where the energy H and $\nabla V = \langle V_1, V_2, V_3\rangle$ is given by

$$H = \ \frac{1}{2}\sum_\ell\sum_i |p_i^\ell|^2 + \frac{1}{2}\sum_{\ell\neq k} V\big(q^\ell - q^k\big).$$

Here, $V : \mathbb{R}^d \to \mathbb{R}$ is an even, nonnegative, smooth, compactly supported function where

$$V_i = \frac{\partial}{\partial q_i}V(q).$$

In the system, the total mass, the three components of momentum, and energy are conserved. The question is how to understand in a scaling limit, as time is speeded up by N, and space is rescaled by N^{-1}, the evolution of the local density $\rho(t,x)$, momenta $w = \langle w_1(t,x), w_2(t,x), w_3(t,x)\rangle$, and energy $e(t,x)$, that is the 'hydrodynamic flow' of the system.

One can form the scaled empirical measures corresponding to these quantities:

$$\begin{aligned}\xi_t^0 &= \frac{1}{N^3}\sum_\ell \delta(\frac{q^\ell(Nt)}{N})\\ \xi_t^i &= \frac{1}{N^3}\sum_\ell \delta(\frac{q^\ell(Nt)}{N})p_i^\ell(Nt) \quad \text{for } i=1,2,3\\ \xi_t^4 &= \frac{1}{N^3}\sum_\ell \delta(\frac{q^\ell(Nt)}{N})h^\ell(Nt)\end{aligned}$$

where energy of the ℓth particle is

$$h^\ell(Nt) \;=\; \frac{1}{2}|p^\ell(Nt)|^2 + \frac{1}{2}\sum_k V\big(q^\ell(Nt) - q^k(Nt)\big).$$

On the torus $N\mathbb{T}^3$, one can define a 5 parameter family of canonical Gibbs point measures $\mu^N_{\rho,w,\beta}$, corresponding to density ρ, velocity w, and inverse temperature β, which are invariant for the dynamics: Here, $L = \rho N^3$ points $\{q^\ell\}$ are distributed in $N\mathbb{T}^3$ according to joint density, with respect to Lebesgue measure,

$$\frac{1}{Z}\exp\big\{-\frac{\beta}{2}\sum_{\ell\neq k} V(q^\ell - q^k)\big\}$$

where Z is a normalization. The momenta $\{p^\ell\}$ are i.i.d. vectors with independent $\{N(w_i,\beta^{-1}) : i = 1,2,3\}$ components, and also independent of the positions $\{q^\ell\}$. An infinite volume limit of these measures, as $N\uparrow\infty$, might be taken under some conditions, e.g. Dobrushin-Lanford-Ruelle specifications.

The goal is to show limits $\xi_t^j \to y^i(t,x)du$ for $j = 0,2,3,4$ where $\langle y_i : i = 0,1,2,3,4\rangle$ satisfies an equation

$$\frac{\partial y}{\partial t} + \nabla_x F(y) \;=\; 0.$$

Here F is a 5×3 matrix, and the equation is Euler's equation. The rigorous passage to such a limit is open! In computing $\frac{\partial}{\partial t}\xi^i$, one has to 'close' the expressions in terms of functions of the empirical measures. Formally, one can do this and derive the form of F, subject to certain ansatz.

For instance, let us derive the equation for the formal limiting 'local average' density $y_0 = \rho(t,x)$ in terms of the formal limiting 'local average' momentum $\langle y^1, y^2, y^3\rangle = w(t,x)$. With respect to a test function G,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{N^3}\sum_\ell G(\frac{q^\ell(Nt)}{N}) &= \frac{1}{N^3}\sum_\ell \nabla G(\frac{q^\ell(Nt)}{N})\cdot p^\ell(Nt)\\ &\sim \int_{\mathbb{T}^3}\big[\nabla G(x)\cdot w(t,x)\big]\rho(t,x)dx.\end{aligned}$$

Then,

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho w) = 0.$$

To derive an equation for $w(t,x)$, however, we will need to understand how to average time-dependent quantities involving nonlinear terms such as

$$\frac{1}{N^3}\sum_\ell p_i^\ell(Nt)p_j^\ell(Nt) \;\text{ and }\; \frac{1}{N^3}\sum_{\ell,k\neq k}\big(q^\ell(Nt) - q^k(Nt)\big)\frac{\partial}{\partial q_i}V(q^\ell(Nt) - q^k(Nt)).$$

One would expect that these quantities could be replaced by space averages with respect to an infinite volume local equilibrium measure.

The difficulty is that such an ergodic theorem for the purely deterministic coupled ODE dynamics to 'close' the equation is not available. Perhaps, the best result is in [8] where a small amount of noise is added to the dynamics to make it into a reasonable Markov process where local averaging can be done, and a rigorous limit can be proved.

In this sense, a motivation for our study of stochastic interacting particle systems is that the randomness of the microscopic motions may allow for suitable ergodic theorems. In a sense, to quote S.R.S. Varadhan, instead of 'approximating an exact problem', we will try to solve 'exactly an approximate problem'.

Exercise 2.1. With the ansatz, in terms of the canonical measures, that

$$p_i^\ell(Nt)p_j^\ell(Nt) \sim w_i(t,x)w_j(t,x) + 1(i=j)\beta^{-1}(t,x)$$

and the 'pressure'

$$\begin{aligned} &P_{j,i}(\rho,\beta^{-1}) \\ &\quad = \lim_{N\to\infty} E_{\mu^N(\rho,w,\beta^{-1})}\Big[\frac{-1}{2N^3}\sum_{\ell,k=1}^N \big(q_j^\ell(Nt)-q_j^k(Nt)\big)\frac{\partial}{\partial q_i}V(q^\ell(Nt)-q^k(Nt))\Big], \end{aligned}$$

derive formally that the equation for $w(t,x)$ is given by

$$\begin{aligned} &\frac{\partial}{\partial t}w_i(t,x) + \frac{\partial}{\partial x_i}\beta^{-1}(t,x) \\ &\qquad + \sum_{j=1}^3 \frac{\partial}{\partial x_j}w_j(t,x)w_i(t,x) + \sum_{j=1}^3 \frac{\partial}{\partial x_j}P_{j,i}\big(\rho(t,x),\beta^{-1}(t,x)\big) = 0. \end{aligned}$$

We note that the last equation for the energy $y^5 = e$ can be seen to be

$$\frac{\partial}{\partial t}e + \nabla\cdot\big[(e+\beta^{-1})w + P(\rho,\beta^{-1})w\big] = 0.$$

3. Simple exclusion processes

We now consider particles on $\mathbb{T}_N^d$ with the minimal interaction that no particle can jump onto another. Accordingly, a configuration of occupation numbers η_t belongs to state space $\Omega = \{0,1\}^{\mathbb{T}_N^d}$ where $\eta_t(x) = 0$ or 1 depending on whether $x \in \mathbb{T}_N^d$ is empty or occupied at time $t \geq 0$. Informally, η_t updates in that each particle is a continuous time random walk carrying an exponential 1 clock. When a clock rings, the particle tries to displace with skeleton jump probability $p(\cdot)$. However, if the site chosen is already occupied, the jump is suppressed, and the clock resets.

More formally, infinitessimally η_t can change to $\eta_t^{x,x+y}$ when a particle jumps from x displaces by increment y where

$$\eta^{a,b}(z) = \begin{cases} \eta(b) & \text{when } z = a \\ \eta(a) & \text{when } z = b \\ \eta(z) & \text{when } z \neq x,y \end{cases}$$

with rate $\eta(x)(1-\eta(x+y))p(y)$. The 'exclusion' factor '$\eta(x)(1-\eta(x+y))$' is 1 exactly when x is occupied and the destination $x+y$ is unoccupied; otherwise, it is 0.

The generator of the process η_t is given by

$$(Lf)(\eta) \;=\; \sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{T}_N^d} \eta(x)(1-\eta(x+y))p(y)\big[f(\eta^{x,x+y})-f(\eta)\big]$$

for bounded functions $f:\Omega\to\mathbb{R}$.

In the following, we will assume that p is finite-range, that is $p(z)=0$ for $|z|>R$ for some R. To check calculations, it may be helpful to assume that p is nearest-neighbor, that is when the range $R=1$.

When p is symmetric, the process is called the 'symmetric simple exclusion process'. When p is asymmetric, η_t is termed the 'asymmetric exclusion process'.

There is a simplification of the form of L when p is symmetric. Namely, since $\eta^{x,x+y}=\eta^{x+y,x}$,

$$\begin{aligned}(Lf)(\eta) \;&=\; \frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{T}_N^d}\big\{\eta(x)(1-\eta(x+y))+\eta(x+y)(1-\eta(x))\big\}p(y)\\ &\qquad\qquad\times\big[f(\eta^{x,x+y})-f(\eta)\big]\\ &=\; \frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{T}_N^d} p(y)\big[f(\eta^{x,x+y})-f(\eta)\big].\end{aligned}$$

The last line follows as the term in curly braces equals $|\eta(x)-\eta(x+y)|=1$ exactly when the difference in square brackets vanishes.

Let ν_α^N be the product measure on $\mathbb{T}_N^d$ with Bernoulli marginals with success probability α.

Proposition 3.1. *The measures $\{\nu_\alpha^N : 0\le\alpha\le 1\}$ are invariant measures for η_t.*

Proof. First note that $f(\eta^{x,x+y})-f(\eta)$ vanishes if $\eta(x)=\eta(x+y)$. Hence, we may write

$$(Lf)(\eta) \;=\; \sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{T}_N^d}\eta(x)p(y)\big[f(\eta^{x,x+y})-f(\eta)\big],$$

where we dropped the factor '$1-\eta(x+y)$'.

Note also, under the change of measure $\zeta=\eta^{x,y}$, which exchanges values $\eta(x)$ and $\eta(y)$, the measure ν_α^N remains the same. Hence, for bounded functions f,g, the term

$$\begin{aligned}&E_{\nu_\alpha^N}\big[g(\eta)\eta(x)(1-\eta(x+y))p(y)f(\eta^{x,x+y})\big]\\ &\qquad = \; E_{\nu_\alpha^N}\big[g(\eta^{x,x+y})\eta(x+y)(1-\eta(x))p(y)f(\eta)\big].\end{aligned}$$

Hence, by collecting terms, with simple manipulation,

$$E_{\nu_\alpha^N}\big[gLf\big] \;=\; \sum_{x,y\in\mathbb{T}_N^d} E_{\nu_\alpha^N}\big[(L^*g)f\big],$$

where the ν_α^N-adjoint L^* is seen as the exclusion generator with single particle jump rate $q(z)=p(-z)$.

Now, since $L^*\mathbf{1}=0$, by inspection (here $\mathbf{1}$ is the constant function 1), we have that $E_{\nu_\alpha^N}[Lf]=0$ for all bounded f. This shows ν_α^N is invariant. □

We remark this proof also shows that ν_α^N is reversible when p is symmetric. Also, when there are exactly K particles in the system, $\nu_\alpha^N(\cdot|\sum_{x\in\mathbb{T}_N^d}\eta(x)=K)$ is the unique 'canonical' invariant measure.

Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a continuous function. We will denote by $\nu^N_{\rho_0(\cdot)}$ as the product measure with Bernoulli marginal at site x with success probability $\rho_0(x/N)$.

Exercise 3.2. Suppose p is symmetric. Show that the space of linear combinations of occupation variables $\eta(x)$ for $x \in \mathbb{T}^N_d$ remains invariant under action by generator L. In fact, the space of linear combinations of $\prod_{j=1}^n \eta(x_j)$ for $\{x_j \in \mathbb{R}^N_d : 1 \leq j \leq n\}$ is closed. Remark: This is a form of *duality*. Hint: First, compute the action on the variable $\eta(x)$.

4. Martingales and Markov chains

Recall that a martingale M_t corresponding to sigma-fields $\mathcal{F}_t$ is a random process, adapted to the filtration $\{\mathcal{F}_t\}$, which satisfies

$$E[M_t|\mathcal{F}_s] \ = \ M_s \ \text{ and } \ E|M_t| < \infty$$

for all $t \geq s \geq 0$.

Exercise 4.1. Let $N(t)$ be a Poisson process with rate λ. Then, $M_t = N(t) - \lambda t$ is a martingale with respect to the 'natural' sigma-fields $\mathcal{F}_t = \sigma\{N_u : u \leq t\}$. Also, $Q_t = M_t^2 - \lambda t$ is also a martingale with respect to $\mathcal{F}_t$.

Let X_t be a Markov process on a countable state space Ω. Let $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a twice continuously differentiable function whose first and second partial time derivatives are uniformly bounded. Define

$$\begin{aligned} M_t^F &= F(t, X_t) - F(0, X_0) - \int_0^t (\frac{\partial}{\partial_s} + L)F(s, X_s)ds \\ N_t^F &= (M_t^F)^2 - \int_0^t (LF^2)(s, X_s) - 2F(s, X_s)(LF)(s, X_s)ds. \end{aligned}$$

These two processes, which we show below are martingales, will be very useful in our stochastic analysis of Markov systems. Often, the corrector term,

$$\langle M_t^F \rangle \ = \ \int_0^t (LF^2)(s, X_s) - 2F(s, X_s)(LF)(s, X_s)ds$$

is referred to as the 'quadratic variation' of the martingale M_t^F.

Proposition 4.2. *With respect to natural sigma-fields $\mathcal{F}_t = \sigma\{X_u : u \leq t\}$, both M_t^F and N_t^F are martingales.*

Proof. We will show that M_t^F is a martingale when F does not depend on time. Generalizations and verification of N_t^F as a martingale are left to the reader. We need only show that

$$E\big[F(X_t)|\mathcal{F}_s\big] - F(X_s) - \int_s^t E\big[(LF)(X_u)|\mathcal{F}_s\big]du \ = \ 0.$$

Now, $E[F(X_t)|\mathcal{F}_s] = P_{t-s}F(X_s)$ and $E[(LF)(X_u)|\mathcal{F}_s] = P_{u-s}(LF)(X_s)$ from the Markov property. From the forward equation, the derivative of the left-side of the above display in t equals

$$P_{t-s}(LF)(X_s) - P_{t-s}(LF)(X_s) \ = \ 0.$$

At time $t = s$, the left-side also vanishes. This concludes the proof. □

Exercise 4.3. Complete the proof of Proposition 4.2. Hint: With respect to M_t^F, we need to show

$$E\big[F(t,X_t)|\mathcal{F}_s\big] - F(s,X_s) = \int_s^t E\big[\big(\frac{\partial}{\partial s} + L\big)F(u,X_u)|\mathcal{F}_u\big]ds.$$

When $t = s$, the relation holds. Hence, if one shows the derivatives with respect to t match, that is

$$\frac{\partial}{\partial t}E\big[F(t,X_t)|\mathcal{F}_s\big] = P_{t-s}\big(\frac{\partial}{\partial t}F\big)(t,x)|_{x=X_s} + P_{t-s}(LF)(t,x)|_{x=X_s}.$$

5. Sketch of the hydrodynamics for simple exclusion processes

Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a continuous function, and let $\mu^N = \nu^N_{\rho_0(\cdot)}$ be a local equilibrium sequence with respect to ρ_0.

Our goal now is to analyze the asymptotic behavior of the empirical measure with respect simple exclusion process η_t,

$$\pi^N_{v(N)t} \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\eta_t(x)\delta_{x/N}$$

in a time scale $v(N)$ to be chosen later. Instead of computing the mean and variance, as with independent particles, we will use the martingale formulation. In particular, the variance with respect to the exclusion interaction is not so easy to handle as before.

To understand the main ideas, let $G : \mathbb{T}^d \to \mathbb{R}$ be a smooth function, not depending on time. Treating

$$\langle G, \pi^N_{v(N)t}\rangle \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x)$$

as a function F on the state space Ω, we obtain a microscopic evolution equation:

$$\langle G, \pi^N_{v(N)t}\rangle \;=\; \langle G, \pi^N_0\rangle + \int_0^{v(N)t} L\langle G, \pi^N_s\rangle ds + M^N_t$$

where M^N_t is a martingale (cf. Proposition 4.2) with quadratic variation

$$\langle M^N_t\rangle \;=\; \int_0^{v(N)t} L\big(\langle G,\pi^N_s\rangle\big)^2 - 2\langle G,\pi^N_s\rangle\big(L\langle G,\pi^N_s\rangle\big)ds.$$

Now, the martingale is negligible in the $N\uparrow\infty$ limit: We compute that

$$\begin{aligned}
&\mathbb{E}_{\mu^N}\big[(M^N_t)^2\big] \qquad\qquad (5.1)\\
&= \int_0^{v(N)t}\frac{1}{N^{2(d+1)}}\sum_{x,y\in\mathbb{T}^d_N}\big(\nabla^N_{x,x+y}G\big)^2 p(y)\eta_s(x)(1-\eta_s(x+y))\\
&\le \frac{v(N)t}{N^{2(d+1)}}\sum_{x,y\in\mathbb{T}^d_N}\big(\nabla^N_{x,x+y}G\big)^2 p(y)\\
&= O(v(N)N^{-d-2}).
\end{aligned}$$

In the last line, we have used that the occupation variables are bounded by 1 and that the jump probabilities are finite-range.

A calculation shows that

$$\begin{aligned} &L\langle G, \pi_s^N\rangle \qquad (5.2)\\ &= \frac{1}{N^{d+1}} \sum_{x,y\in\mathbb{T}_N^d} \eta_s(x)(1-\eta_s(x+y))p(y)\big[\nabla^N_{x+y,x}G\cdot(\eta_s(x)-\eta_s(x+y))\big]\\ &= \frac{1}{N^{d+1}} \sum_{x,y\in\mathbb{T}_N^d} \eta_s(x)(1-\eta_s(x+y))p(y)\big[\nabla^N_{x+y,x}G\big] \end{aligned}$$

where $\nabla^N_{u,v}G = N[G(u/N) - G(v/N)] \sim (u-v)\cdot\nabla G(v/N)$.

Exercise 5.1. Verify the form of the quadratic variation given in (5.1).

5.1. **Symmetric case.** When p is symmetric, a further summation by parts is possible and we obtain in this case that

$$\begin{aligned} L\langle G, \pi_s^N\rangle &= \frac{1}{2N^{d+1}} \sum_{x,y\in\mathbb{T}_N^d} \big|\eta_s(x)-\eta_s(x+y)\big|p(y)\nabla^N_{x+y,x}G\\ &= \frac{1}{2N^{d+2}} \sum_{x,y\in\mathbb{T}_N^d} p(y)\eta_s(x)\triangle^N_{x,y}G \end{aligned}$$

where $\triangle^N_{x,y}G = N^2[G(x+y/N) - 2G(x/N) + G(x-y/N)]$.

Now, $\triangle^N_{x,y}G = \triangle_C G(x/N) + o(1)$ where

$$\triangle_C = \sum_{1\le i,j\le d} C_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}, \quad \text{and covariances } C_{i,j} = \sum_{z\in\mathbb{T}_N^d} z_i z_j p(z).$$

Hence, if $v(N) = N^2$, we have that

$$\langle G, \pi^N_{v(N)t}\rangle = \langle G, \pi^N_0\rangle + \frac{1}{2N^2}\int_0^{v(N)t}\langle\triangle_C G, \pi^N_s\rangle ds + M^N_t + o(1).$$

Putting these estimates together, for symmetric p, we have 'closed' the equation:

$$\langle G, \pi^N_{v(N)t}\rangle = \langle G, \pi^N_0\rangle + \frac{1}{2}\int_0^{t}\langle\triangle_C G, \pi^N_{v(N)s}\rangle ds + o(1).$$

This suggests in the $N\uparrow\infty$ limit that the empirical measures $\pi^N_{N^2t}$ converge in the weak sense to a solution of the Heat equation $\partial_t\rho = (1/2)\triangle_C\rho$.

The goal of the next lecture is to make precise this statement for symmetric simple exclusion.

5.2. **Drift case.** When p is asymmetric, say $m = \sum_z zp(z) \neq 0$, as in the independent particle model, we should choose $v(N) = N$. But, one cannot 'close' the equation. One has to deal with the term

$$\frac{N}{N^{d+1}} \sum_{x,y\in\mathbb{T}_N^d} \eta_s(x)(1-\eta_s(x+y))p(y)\big[\nabla^N_{x+y,x}G\big]$$

composed of 'two-point' functions $\eta(x)\eta(x+y)$. In the limit, such a term due to 'local averaging' should be replaced by a quadratic function of the empirical density.

Formally, one would obtain a form of Burger's equation

$$\partial_t + m\cdot\nabla\rho(1-\rho) = 0.$$

There is much work on such hyperbolic mass conservation laws. In particular, there may be several solutions even starting from smooth initial data. It will turn out that the solution found by hydrodynamics is the unique 'entropy' or 'vanishing viscosity' solution. We will discuss the $m \neq 0$ case in a subsequent lecture.

5.3. **Asymmetric, mean-zero case.** In the final case when p is asymmetric, but mean-zero, that is $p(z) \neq p(-z)$ for some z, but $\sum_z zp(z) = 0$, there are other complications. The system is referred to as a 'non-gradient' model. Although, we should speed up time by $v(N) = N^2$, a second summation-by-parts as in the symmetric situation cannot be done. However, multiplying (5.2) by N^2, we have

$$\frac{N^2}{2N^{d+1}} \sum_{x \in \mathbb{T}_N^d} \nabla G(x/N) \cdot \sum_{y \in \mathbb{T}_N^d} \big\{ \eta_s(x)(1-\eta_s(x+y))yp(y) - \eta_s(x+y)(1-\eta_s(x))yp(-y) \big\}.$$

Unless p is symmetric, one cannot rewrite the expression in curly braces as the exact difference of a function f and its translate.

Let $\{e_i\}$ be the standard basis in $\mathbb{Z}^d$. It can be shown that the sum over y dotted with e_i can be approximated by a difference $\sum_{j=1}^d a_{i,j}(\eta(x)) - a_{i,j}(\eta(x + e_j))$, a 'gradient'. The function $a_{i,j}$ depends on p. With respect to a certain 'homogenization' $\bar{a}_{i,j}$ of $a_{i,j}$, the hydrodynamic equation can then be written down as a nonlinear Heat equation,

$$\partial_t \rho = (1/2) \sum_{i=1}^d \frac{\partial}{\partial_{x_i, x_j}} \bar{a}_{i,j}(\rho(t, \theta)).$$

We refer to [13] and [3] for more discussion and details.

6. Notes

For further discussion of hydrodynamics of deterministic evolutions see [12][Part 1] and [9]. See also [13] which elaborates more on the exercise 2.1.

The material on Martingales can be found in [1] for instance, and other places. Similar treatments of the hydrodynamics of simple exclusion can be found in [3] and [13].

The simple exclusion process, introduced in [11] (see [2] for a retrospective), has many properties which make it amenable to calculation. It has proved to be a versatile model, which can be defined on very general graphs, in applications and theoretical studies as a web search reveals. See [5], [6], [7], [10] for detailed studies.

As one can see in the sketch of the hydrodynamics for symmetric simple exclusion processes, one can weaken requirements on the initial measure. In fact, what is needed is a guarantee of a law of large numbers at time 0. The concept of 'very weak local equilibrium' given below is sufficient and somewhat general.

Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a function. We say that a sequence of probability measures μ^N on $\mathbb{T}_N^d$ is a 'very weak local equilibrium' according to profile ρ_0 if

$$\lim_{N \uparrow \infty} \Big[\Big| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(x/N)\eta(x) - \int_{\mathbb{T}^d} G(u)\rho_0(u)du \Big| \Big] \;=\; 0.$$

for all bounded, continuous $G : \mathbb{T}^d \to \mathbb{R}$.

We remark that μ^N may be degenerate, that is supported on a single configuration, and that the sequence $\{\mu^N\}$ may consist of deterministic configurations which satisfy the law of large numbers in the definition above.

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