LECTURE 1: PRELIMINARIES AND HYDRODYNAMICS OF INDEPENDENT RANDOM WALKS

SUNDER SETHURAMAN

Before discussing a basic example, illustrating possibilities in the study of 'hydrodynamics of stochastic particle systems', we recall some basic notions in Markov chains.

1. Markov chains

1.1. **Construction.** A family of random variables $\{X_n : n \geq 0\}$ taking values on a countable state space E is called a 'discrete time Markov chain' if the 'stationary Markov property' is satisfied:

$$\begin{aligned} P(X_n = x_n | X_0 = x_0, \ldots, X_m = x_m) &= P(X_n = x_n | X_m = x_m) \\ &= P(X_{n-m} = x_n | X_0 = x_m) \end{aligned}$$

for all $x_0, \ldots, x_m \in E$ and $n > m \geq 0$. When $n = 1$ and $m = 0$, the last quantity on the right-side represents a 'transition probability' of the Markov chain, denoted $p(x, y) = P(X_1 = y | X_0 = x)$. The n-step probability $p^{(n)}(x, y) = P(X_n = y | X_0 = x)$ satisfies a recurrence, $p^{(n+1)}(x, y) = \sum_{z \in E} p^{(n)}(x, z) p(z, y)$.

We now construct a 'continuous time Markov chain' on E with 'skeleton' $\{X_n : n \geq 0\}$ and transition probability vanishing on the diagonal, that is $p(x, x) = 0$ for all $x \in E$. Let $\{\lambda_x : x \in E\}$ be a collection of positive numbers, and let $\{W_n : n \geq 0\}$ be a collection of independent identically distributed exponential random variables with rate 1, independent of the skeleton discrete time chain. Define now the process $\{Z_t : t \geq 0\}$ as follows: Initially, $Z_0 = X_0 \in E$. After time $\lambda_{X_0}^{-1} W_0$, the process jumps to value X_1, and after a subsequent time $\lambda_{X_1}^{-1} W_1$, the process jumps to value X_2, and so on. Let $T_k = \sum_{i=0}^{k} \lambda_{X_i}^{-1} W_i$ for $k \geq 0$. Then,

$$Z_t = \begin{cases} x & \text{for } 0 \leq t < T_0 \\ X_1 & \text{for } T_0 \leq t < T_1 \\ \vdots & \vdots \\ X_n & \text{for } T_{n-1} \leq t < T_n \end{cases}$$

for $0 \leq t < T_\infty = \lim_{n \uparrow \infty} T_n$. Sufficient conditions for $T_\infty = \infty$ include the cases if the state space E is finite, or if $\sup_{x \in E} \lambda_x < \infty$. We will assume from now on that the process is 'regular', that is $T_\infty = \infty$, so that the process Z_t is defined for all time $t \geq 0$.

One can show that the 'continuous time' chain $\{Z_t : t \geq 0\}$ satisfies the stationary Markov property, which in this context is equivalent to

$$\begin{aligned} P(Z_t = y | Z_{t_0} = x_0, \ldots, Z_{t_m} = x_m, Z_s = x) &= P(Z_t = y | Z_s = x) \\ &= P(Z_{t-s} = y | Z_0 = x) \quad (1.1) \end{aligned}$$

for all $x, y, x_0, \ldots, x_m \in E$, $0 \leq t_0 < \cdots t_m < s < t$ and $m \geq 0$.

Conversely, given a process $\{Z_t : t \geq 0\}$ satisfying (1.1) and also the 'jump property' that there exists a sequence of strictly increasing stopping times $\{T_n : n \geq 0\}$ such that $T_0 > 0 = T_{-1}$ and Z_t is constant on intervals $[T_n, T_{n+1})$ and $Z_{T_n^-} \neq Z_{T_n}$ for $n \geq 0$, one can determine a unique skeleton discrete time Markov chain $\{X_n : n \geq 0\}$ and positive jump parameters $\{\lambda_x : x \in E\}$ such that $X_n = Z_{T_n}$, $p(x, y) = P(Z_{T_{n+1}} = y | Z_{T_n} = x)$, and $T_n - T_{n-1}$ are independent exponentials with rates λ_{X_n} for $n \geq 0$. Chains with the same skeleton and jump parameters have the same joint distributions. The condition $Z_{T_n^-} \neq Z_{T_n}$ ensures p vanishes on the diagonal.

1.2. **Generators, Chapman-Komogorov equations.** In the discrete time setting, we can define the transition operator $P = (p(x, y) : x, y \in E)$, which is a matrix when E is finite. Then, the nth powers give the nth step probabilities, $P^n(x, y) = P(X_n = y | X_0 = x)$.

Computing the t-time probabilities for the continuous time Markov chain Z_t is more complicated. Define the transition probability

$$P_t(x, y) \;=\; P(Z_t = y | Z_0 = x).$$

Then, by the Markov property we have

$$P_{t+s}(x, y) \;=\; \sum_{z \in E} P_t(x, z) P_s(z, y)$$

or in terms of operators $P_{t+s} = P_t P_s$, the semigroup property.

Given regularity of the process, the transition functions are differentiable in time, and satisfy the 'first jump' relation

$$\begin{aligned} P_t(x, y) &= P(Z_t = y, T_0 > t | Z_0 = x) + P(Z_t = y, T_0 \leq t | Z_0 = x) \\ &= 1(x = y) e^{-\lambda_x t} + \int_0^t \lambda_x e^{-\lambda_x s} \sum_{z \neq x} p(x, z) P_{t-s}(z, y), \end{aligned}$$

from which the backward equation

$$\begin{aligned} \frac{d}{dt} P_t(x, y) &= \sum_{z \in E} \lambda_x p(x, z)[P_t(z, y) - P_t(x, y)] \\ P_0(x, y) &= 1(x = y). \end{aligned}$$

Similarly, from a decomposition of the jump time just before time t, one has the forward equation

$$\frac{d}{dt} P_t(x, y) \;=\; \sum_{z \in E} P_t(x, z) \lambda_z p(z, y) - P_t(x, y) \lambda_y.$$

Exercise 1.1. Review this derivation. To reduce notation, one can try in finite-state space first.

Define the operator

$$L(x, y) \;=\; \begin{cases} \lambda_x p(x, y) & \text{for } y \neq x \\ -\lambda_x & \text{for } y = x. \end{cases}$$

Then, neatly expressed, the backward and forward equations become

$$\frac{d}{dt} P_t \;=\; L P_t \quad \text{and} \quad \frac{d}{dt} P_t \;=\; P_t L.$$

Also, $\lim_{t\downarrow 0} t^{-1}[P_t - I] = L$ and $P_t(x,y) = \delta_{x,y} + tL(x,y) + o(t)$.

When the space E is finite, L is a 'generator' matrix, that is $L(x,y) \geq 0$ for $x \neq y$ and $L(x,x) = -\sum_{y\neq x} L(x,y)$, and by solving the ODE's, one obtains $P_t = e^{tL}$ which can be computed in some cases. More generally, P_t can be understood in terms of the Hille-Yosida formulation.

Let $f : E \to \mathbb{R}$ be a bounded function on the state space. In the discrete time case, define $Pf(x) = \sum_{y\in E} p(x,y)f(y)$, which is the conditional expectation of $f(X_1)$ given $X_0 = x$. Then, $P^n f(x) = \sum_{y\in E} p^{(n)}(x,y)f(y)$ is the conditional expectation of $f(X_n)$ given $X_0 = x$, where $p^{(n)}(x,y)$ is the n-fold convolution, or nth step transition probability.

In the continuous case, define $P_t f(x) = \sum_{y\in E} P_t(x,y)f(y)$ which is the conditional expectation of $f(Z_t)$ given $Z_0 = x$. In this framework, the generator L is often expressed in terms of its action on f:

$$\begin{aligned}(Lf)(x) &= \sum_{y\in E} L(x,y)[f(y) - f(x)] \\ &= \sum_{y\in E} \lambda_x p(x,y)[f(y) - f(x)].\end{aligned}$$

1.3. **Invariant measures.** Let μ be a probability measure on E. In the discrete time situation, define $\mu P(x) = \sum_{y\in E} \mu(y)p(y,x)$. Hence, we see that μP^n is the distribution at time n when the initial state is distributed according to μ.

In the continuous model, define

$$\mu P_t(x) = \sum_{y\in E} \mu(y)P_t(y,x), \quad \text{and} \quad \mu L(x) = \sum_{y\in E} \mu(y)L(y,x).$$

We say that μ is an 'invariant measure' for discrete time chains if $\mu P = \mu$, and for continuous time chains if $\mu P_t = \mu$ for all $t \geq 0$. We also say that μ is a 'reversible' invariant measure in discrete time chains if $\mu(x)p(x,y) = \mu(y)p(y,x)$ for all $x, y \in E$. In continuous time chains, μ is 'reversible' when P_t is self-adjoint in $L^2(\mu)$, that is $\sum_{x\in E} \mu(x)f(x)P_t(x,y)g(y) = \sum_{x\in E} \mu(x)g(x)P_t(x,y)f(y)$, or in terms of the inner product on $L^2(\mu)$, $\langle f, P_t g\rangle_\mu = \langle P_t f, g\rangle_\mu$ for all $f, g \in L^2(\mu)$.

One can verify that in the continuous time setting that μ being invariant is equivalent to $\mu L = 0$, and μ being reversible is equivalent to $\mu(x)L(x,y) = \mu(y)L(y,x)$ for all $x, y \in E$, or in terms of inner products $\langle f, Lg\rangle_\mu = \langle Lf, g\rangle_\mu$ for all $f, g \in L^2(\mu)$.

Exercise 1.2. Review these equivalences.

In the finite state space case, invariant measures always exist, and if the skeleton chain can reach every state from any state in finite time, that is p is irreducible, the invariant measure is unique. However, in the countable state case there may be no invariant measures.

Reversibility has the following interesting implication. Fix a time $t > 0$, and consider $R_s = Z_{t-s}$ for $0 \leq s \leq t$. Suppose that initially Z_0 is distributed according to an invariant measure μ. Then, it can be seen that R_s is a continuous time Markov chain with (in geneneral nonstationary) transition probability $Q_t(x,y) = (\mu(y)/\mu(x))P_t(y,x)$. In particular, when μ is reversible, $Q_t(x,y) = P_t(x,y)$ and in this case the 'forward in time' and 'backward in time' chains have the same distribution!

We remark that it is sometimes easier to find directly a reversible measure and therefore an invariant measure by checking the reversibility conditions.

1.4. **Examples.** We will content ourselves for the moment with two basic continuous time examples, the two-state Markov chain, and random walk.

Example 1.3. Let $E = \{0, 1\}$ correspond to states 'on' and 'off', or sometimes 'empty' and 'occupied'. Here, state 0 can transition to state 1 and vice versa. Let λ_0 and λ_1 be the corresponding jump rates. The skeleton chain probabilities are $p(0, 1) = p(1, 0) = 1$. The generator matrix is

$$L \;=\; \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix}.$$

and correspondingly, it is an exercise in diagonalization to compute $P_t = e^{tL}$ and to find the unique invariant measure.

Exercise 1.4. Compute the invariant measure and P_t. Is it reversible?

Example 1.5. We define the Poisson process with parameter α. Let $E = \mathbb{N} = \{0, 1, 2, \ldots\}$, the whole numbers. Let $\lambda_x \equiv \alpha$. The skeleton chain is deterministic where transitions are to the nearest right site: $p(x, x + 1) = 1$ for $x \geq 0$. Then, with $Z_0 = 0$, the continuous time chain Z_t counts the number steps made up to time $t \geq 0$. The trajectories are step functions, which are right-continuous, with left limits.

Exercise 1.6. Compute the backward/forward equation, and show that the transition probability is in 'Poisson' form $P_t(0, x) = P_t(y, y + x) = e^{-\alpha t}(\alpha t)^x/x!$.

Example 1.7. Let $E = \mathbb{T}_N^d$ the d-dimensional torus where $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, or integers modulo N. Let also $\lambda_x \equiv 1$, and p be a finite-range, translation-invariant transition probability: For all $x, y \in E$, $p(x, y) = 0$ if $|x - y| \geq R$ some $R < \infty$, and $p(x, y) = p(0, y - x) =: p(y - x)$. We will also assume that p is irreducible. For instance, the nearest-neighbor, symmetric case is one possibility.

Then, $L(x, y) = p(x, y)$ for $x \neq y$ and $L(x, x) = -\sum_{y \neq x} L(x, y)$. In particular, the uniform distribution $\mu(x) \equiv N^{-1}$ is the unique invariant measure. Moreover, μ is reversible exactly when $p(x) = p(-x)$ for all $x \in E$.

Now let R_t be the number of jumps before time t. Since the jump rates are all 1, we see that R_t is a Poisson process with rate 1. In particular, the transition probability

$$P_t^N(y - x) := P_t(x, y) \;=\; E\big[p^{(R_t)}(x, y)\big] \;=\; \sum_{n \geq 0} \frac{e^{-t}t^n}{n!} p^{(n)}(x, y).$$

We now consider a sequence of chains $Z_t = Z_t^{(N)}$ on a sequence of torii $\mathbb{T}_N^d$ as $N \geq 1$ increases. Define $m = \sum_{x \in E} x p(x)$ to be the mean displacement of the position. Then, it is not difficult to extablish the following law of large numbers,

$$\lim_{N \uparrow \infty} \frac{Z_{Nt}^{(N)}}{N} \;=\; mt \quad \text{in probability}.$$

Here, $m \in \mathbb{T}^d$ where $\mathbb{T}$ is the unit circle.

Exercise 1.8. Show this LLN by say variance computations. Noting R_t is Poisson with variance t is useful. One can do it also by regeneration, and other methods.

Also, if $m = 0$, let σ be the matrix of covariances, $\sigma_{i,j} = \sum_x x_i x_j p(x)$ for $0 \le i, j \le d$. We have the central limit theorem,

$$\frac{Z^{(N)}_{N^2 t}}{N} \Rightarrow \mathrm{N}(0, \sigma t).$$

Exercise 1.9. There are a few ways to show this CLT. One way is to compute the moment generating function or characteristic function, noting independence of $R^{(N)}_t$ and the displacements.

Finally, when $m = \sum_x xp(x) \neq 0$, we say the walk is asymmetric, and when $m = 0$ and $p(\cdot)$ is not symmetric, we say the walk is mean-zero asymmetric, and when $p(\cdot)$ is symmetric, we say the walk is symmetric.

2. Hydrodynamics of independent random walks

We would like to understand the space-time evolution of the mass in a system of particles with a conservation law. Perhaps the simplest model is that of unlabeled non-interacting random walks on a d-dimensional torus with N locations. When N is large, and one looks at the system from afar, after long times, one can more discern the motion of the bulk of the mass rather than individual components. The goal is to make precise the evolution of the mass in this scale in terms of a continuum equation.

We will be working with a Markov chain on $E = \mathbb{N}^{\mathbb{T}^d_N}$, where we recall $\mathbb{N} = \{0, 1, 2, \ldots\}$, which governs the motion of K independent random walks on $\mathbb{T}^d_N$ moving as in Example 1.7. Since we are interested in the 'mass' of particles, we will consider the occupation numbers at each location on the lattice $\mathbb{T}^d_N$. That is, let Z^i_t be the position of the ith particle at time t. Define

$$\eta_t(x) = \sum_{i=1}^{K} 1(Z^i_t = x).$$

We now observe that the process $\eta_t = \{\eta_t(x) : x \in \mathbb{T}^d_N\}$ is a Markov chain. Indeed, given independence and the Markov property of the individual particle movement, by splitting over all possibilities, the Markov property of η_t can be deduced.

Exercise 2.1. Show this Markov property.

2.1. **Invariant measures and distribution at time $t \ge 0$.**

What are the invariant measures for the process? Since the process η_t, corresponding to K particles, is irreducible, there is a unique invariant measure. It is not so easy to characterize it immediately. We will first consider a certain 'relaxation' which helps in the formulation.

Indeed, let us relax the assumption that there are K particles in the system. If we do not specify the initial number of particles, then η_t is no longer irreducible, since there is no birth or death possible: For instance, a system with 10 particles cannot evolve into one with 20 random walks. Nevertheless, we may specify in nice form several invariant measures for this 'relaxed' system.

Recall the Poisson distribution with parameter α, $q_\alpha(k) = e^{-\alpha}\alpha^k/k!$ for $k \geq 0$. Its moment generating function is given by

$$\sum_{k\geq 0} e^{\lambda k} e^{-\alpha}\frac{\alpha^k}{k!} \;=\; e^{\alpha(e^\lambda - 1)}.$$

For a nonnegative function $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$, define the product measure $\nu^N_{\rho_0(\cdot)}$ on $\mathbb{N}^{\mathbb{T}^d_N}$ by

$$\nu^N_{\rho(\cdot)}(\eta(x) = k) \;=\; q_{\rho_0(x/N)}(k).$$

When $\rho_0(\cdot) \equiv \alpha$ is constant, we denote $\nu^N_{\rho_0(\cdot)} = \nu^N_\alpha$.

The process $\{\eta_t : t \geq 0\}$ belongs to the space of right-continuous paths with left limits in $E = \mathbb{N}^{\mathbb{T}^d_N}$, $D([0,\infty); \mathbb{N}^{\mathbb{T}^d_N})$. We will denote by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the probability measure and expectation with respect to the evolution of the process when initially η_0 is distributed according to μ. On the other hand, E_μ will refer to the expectation with respect to μ on E.

Now, starting from $\nu^N_{\rho(\cdot)}$, we observe that we may calculate the distribution at later times $t > 0$.

Proposition 2.2. *Under $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$, the distribution of η_t is the inhomogeneous product of Poisson measures $\prod_{x\in\mathbb{T}^d_N} q_{\psi_{N,t}(x)}$ where $\psi_{N,t}(x) = E\big[\rho_0(N^{-1}(x - Z^N_t))\big]$ is the expectation with respect to the position of a random walk Z^N_t with rates $p(\cdot)$ starting from the origin at time t.*

That the later time t-distribution would still be a product measure is a feature of the independence of the particles, and is not to be inferred for more general interacting particle systems.

When $\rho_0 \equiv \alpha$ is constant, we immediately arrive at the following characterization.

Corollary 2.3. *The measures ν^N_α are invariant for the Markov chain η_t.*

Proof of Proposition 2.2. We need only compute the moment generating function of η_t.

Write

$$\eta_t(x) \;=\; \sum_{y\in\mathbb{T}^d_N}\sum_{k=1}^{\eta_0(y)} 1(Z^{y,k}_t = x)$$

and, for $\theta : \mathbb{T}^d_N \to \mathbb{R}$,

$$\sum_{x\in\mathbb{T}^d_N}\theta(x)\eta_t(x) \;=\; \sum_{x\in\mathbb{T}^d_N}\sum_{y\in\mathbb{T}^d_N}\sum_{k=1}^{\eta_0(y)}\theta(x)1(Z^{y,k}_t = x) \;=\; \sum_{y\in\mathbb{T}^d_N}\sum_{k=1}^{\eta_0(y)}\theta(Z^{y,k}_t)$$

where $Z^{y,k}_t$ denotes the position at time t of the kth particle initially at location y.

Since particles move independently, and initially there are a Poisson number of particles on each site of the lattice,

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\exp\sum_{x\in\mathbb{T}^d_N}\theta(x)\eta_t(x)\Big] &= \prod_{y\in\mathbb{T}^d_N}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\exp\sum_{k=1}^{\eta_0(y)}\theta(Z_t^{y,k})\Big]\\ &= \prod_{y\in\mathbb{T}^d_N}E_{\nu^N_{\rho_0(\cdot)}}\Big(E\big[\exp\theta(Z_t^{y,1})\big]\Big)^{\eta_0(y)}\\ &= \prod_{y\in\mathbb{T}^d_N}\exp\Big[\rho_0(y/N)\big(E\big[e^{\theta(y+Z_t)}\big]-1\big)\Big]\end{aligned}$$

where $Z_t = Z_t^N$ is the position of a random walk on $\mathbb{T}^d_N$ starting at the origin.

Now,

$$E\big[e^{\theta(y+Z_t)}\big] = \sum_{x\in\mathbb{T}^d_N}P_t^N(x-y)e^{\theta(x)}$$

and

$$E\big[e^{\theta(y+Z_t)}\big]-1 = \sum_{x\in\mathbb{T}^d_N}P_t^N(x-y)(e^{\theta(x)}-1).$$

Hence, after a calculation,

$$E_{\nu^N_{\rho_0(\cdot)}}\Big[\exp\sum_{x\in\mathbb{T}^d_N}\theta(x)\eta_t(x)\Big] = \exp\sum_{y\in\mathbb{T}^d_N}\rho_0(y/N)\sum_{x\in\mathbb{T}^d_N}P_t^N(x-y)\big(e^{\theta(x)}-1\big).$$

Since

$$\begin{aligned}\sum_{y\in\mathbb{T}^d_N}P_t^N(x-y)\rho_0(y/N) &= \sum_{z\in\mathbb{T}^d_N}P_t^N(z)\rho_0(N^{-1}(x-z))\\ &= E\big[\rho_0\big(N^{-1}(x-Z_t^N)\big)\big] = \psi_{N,t}(x),\end{aligned}$$

the proof is finished. □

To come back to our initial question, we now remark that the invariant measure ν^N_α can be decomposed in terms of its restrictions to the sets $\{\eta\in E : \sum_{x\in\mathbb{T}^d_N}\eta(x) = K\}$ for $K\geq 0$, which are closed and irreducible for the motion. Then, each $\nu_{\mathbb{T}^d_N,K} = \nu^N_\alpha(\cdot|\sum_{x\in\mathbb{T}^d_N}\eta(x)=K)$, is the unique invariant measure on the restriction, and does not depend on α. In physics terminology, ν^N_α is the 'grand canonical' measure and $\nu_{\mathbb{T}^d_N,K}$ is the 'canonical' one.

Since the mean of $\eta(x)$ under ν^N_α equals α, it makes sense to call α the 'mass density' of the process. In this way, $\{\nu^N_\alpha : \alpha\geq 0\}$ is a family of invariant measures indexed by density α.

When ρ_0 is not constant, the measures $\nu^N_{\rho_0(\cdot)}$ are examples of 'local equilibrium' measures, as near a continuity of point $u\in\mathbb{T}^d$ of $\rho_0(\cdot)$, they distribute particles near $\lfloor uN\rfloor$ like the invariant measure $\nu^N_{\rho_0(u)}$. One may consider other non-product 'local equilibrium' and 'nonequilibrium' measures, but, as we have seen, these local equilibrium measures allow for some computations.

2.2. **Hydrodynamics.** If we start with a local equilibrium measure $\nu^N_{\rho_0(\cdot)}$, then initially, the means $\{\rho_0(x/N) : x \in \mathbb{T}^d_N\}$ are a discretization of the density 'profile' ρ_0 defined on the continuous space $\mathbb{T}^d$. At a later time t, the means have evolved to $\{\psi_{N,t}(x) : x \in \mathbb{T}^d_N\}$. But, how to understand a macroscopic picture?

The answers depend on the particular time and space scales chosen in the problem. We will think of $\mathbb{T}^d_N$ as embedded in $\mathbb{T}^d$ where grid points on $\mathbb{T}^d_N$ are separated by distance N^{-1}. In this way, a 'macroscopic' point u on $\mathbb{T}^d$ corresponds to the 'microscopic' point $\lfloor uN \rfloor$. As we will see, time should now be appropriately speeded up to see movement of the system. How fast this speed up should be will depend on the structure of the underlying jump probability $p(\cdot)$.

When t is fixed, one can see that $N^{-1}Z^N_t$ converges weakly to 0. Hence, at a continuity point u for $\rho_0(\cdot)$, we have $\lim_{N\uparrow\infty} \psi_{N,t}(\lfloor uN \rfloor) = \rho_0(u)$. So, if we do not speed up time at all, the system does not move.

2.3. **Asymmetric motions.** In the asymmetric setting, $m = \sum xp(x) \neq 0$, since $N^{-1}Z^N_{tN} \to mt$ in probability, we have

$$\lim_{N\uparrow\infty} \sum_{|z/N-mt|\leq\epsilon} P^N_{tN}(z) \;=\; \lim_{N\uparrow\infty} P\Big[\Big|\frac{Z^N_{Nt}}{N} - mt\Big| \leq \epsilon\Big] \;=\; 1.$$

In this case, when the initial profile ρ_0 is continuous,

$$\lim_{N\uparrow\infty} \psi_{N,Nt}(\lfloor uN \rfloor) \;=\; \rho_0(u - mt) \;:=\; \rho(t,u).$$

Therefore, if we speed up time by a factor of N, we see that the density profile has translated by mt. In this sense Nt is referred to as the 'microscopic' time, and t as the 'macroscopic' time. Moreover, the density $\rho(t,u)$ satisfies

$$\partial_t\rho + m\cdot\nabla\rho \;=\; 0. \tag{2.1}$$

This makes sense as individual particles displace an order N microscopic locations at microscopic time Nt.

2.4. **Mean-zero motions.** However, when $m = 0$, particles do not displace as much, but follow the 'square root' law, when say the underlying jump rates are finite-range. In other words, displacements are of order $\sqrt{N}$ at time Nt, or alternatively of order N at times N^2t. The latter version fits in nicely with our space scaling of N^{-1}.

By the central limit theorem for random walks in this case, $N^{-1}Z^N_{N^2t} \Rightarrow \mathrm{N}(0, \sigma t)$, when again ρ_0 is continuous, we have

$$\begin{aligned}\lim_{N\uparrow\infty} \psi_{N,N^2t}(\lfloor Nu \rfloor) \;&=\; \lim_{N\uparrow\infty} \sum_{z\in\mathbb{T}^d_N} P^N_{N^2t}(z)\rho_0\big(N^{-1}(\lfloor Nu \rfloor - z)\big)\\ &=\; \lim_{N\uparrow\infty} E\Big[\rho_0(u - N^{-1}Z^N_{N^2t})\Big] \;=\; \int_{\mathbb{R}^d} \bar\rho_0(x)G_t(u-x)dx.\end{aligned}$$

Here, $\bar\rho_0$ is the periodic extension of ρ_0 with period $\mathbb{T}^d$, and G_t is the Gaussian density with covariance $t\sigma^2$. It follows that $\rho(t,u) := \int_{\mathbb{R}^d} \bar\rho_0(x)G_t(u-x)dx$, as a convolution with the $\bar\rho_0$, satisfies the heat equation

$$\begin{aligned}\partial_t\rho \;&=\; \sum_{1\leq i,j\leq d} \sigma^2_{i,j}\partial^2_{u_i,u_j}\rho\\ \rho(0,u) \;&=\; \rho_0(u).\end{aligned} \tag{2.2}$$

2.5. **Conclusion.** What we have done so far is to derive a macroscopic 'hydrodynamic limit' for the mean values over $x \in \mathbb{T}_N^d$. In the next lecture, we will view the hydrodynamic limit as a full fledged law of large numbers of the empirical measure of particles.

We call the equations (2.1) and (2.2) and their solutions $\rho(t,u)$ as 'hydrodynamic' equations and 'hydrodynamic' solutions for the space-time evolution of the macroscopic density. For independent particles, to summarize, we have proved the following:

Theorem 2.4. *Suppose $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ is continuous. Let $v(N) = N$ if $m \neq 0$ and $v(N) = N^2$ if $m = 0$. Then, starting from the sequence $\nu^N_{\rho_0(\cdot)}$, the density average $\psi_{N,v(N)t}(\lfloor Nu \rfloor)$ at location $\lfloor Nu \rfloor$ and time $v(N)t$ converges to $\rho(t,u)$ which solves equation* (2.1) *if $m \neq 0$ and equation* (2.2) *if $m = 0$.*

3. Notes

The material on Markov chains is a standard treatment based on the development in [3], [10] and [4, Appendix 1].

Recent work on invariant measures and hydrodynamics of systems of independent particles includes [2], [7], [6], [9]. See also [8] for a more general treatment. The first rigorous work on hydrodynamics of independent particle systems, on which the above development rests, is [1]; see also [4, Chapter 1] and [5] for larger discussions.

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Mathematics, University of Arizona, Tucson, AZ 85721
Email address: sethuram@math.arizona.edu