On fluctuations in interacting particle systems

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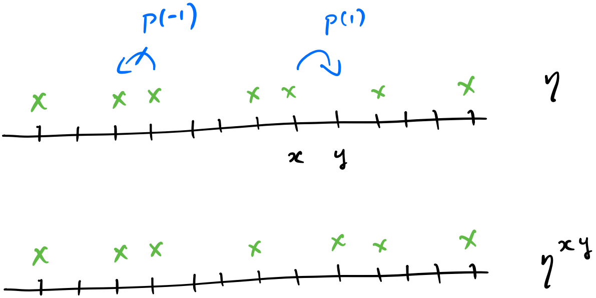
November 10-15, 2025

Symposium of probability and stochastic processes
Universidad de Sonora

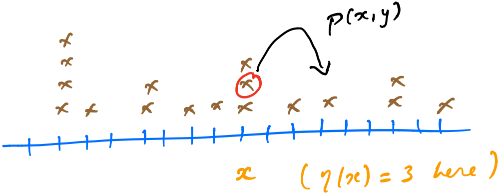
Outline

- Local statistics: Occupation times
- Local statistics: Tagged particles
- Bulk limits: LLN of the ‘bulk’ mass and hydrodynamics
- Bulk limits: Fluctuations of the ‘bulk’ mass
  - Calculations for Exclusion
  - Boltzmann-Gibbs for Zero-range, etc.
  - Discussion of some methods

Exclusion interactions



Zero-range interactions



Hydrodynamic limit

Consider the $d \geq 1$ nearest-neighbor symmetric Exclusion process on $\mathbb{Z}^d$.

–Recall that $\eta_t^N = \eta_{N^2 t}$ with $\theta = 2$, and μ^N is an initial 'local' equilibrium measure associated to $\rho_0 : \mathbb{R}^d \to [0, 1]$.

We have

$$\pi_t^N = \frac{1}{N^d} \sum_x \eta_t^N(x) \delta_{x/N} \Rightarrow \rho(t, u) du$$

where

$$\partial_t \rho = p(e) \Delta \rho.$$

–Here, $p(e) = 1/2d$.

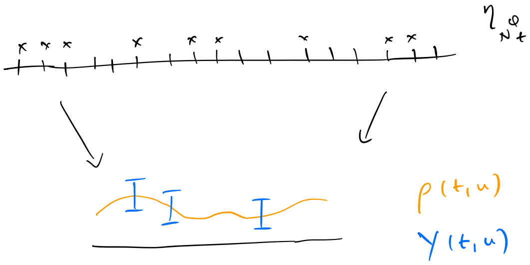
Fluctuations from the hydrodynamic limit

One might ask about the scale of the 'errors' in the hydrodynamic limit.

Define the fluctuation field

$$Y_t^N = \frac{1}{N^{d/2}} \sum_x \Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N(x)] \Big) \delta_{x/N}.$$

$d=1$



How does Y evolve?

We may look at the evolution of

$$Y_t^N(J) = \frac{1}{N^{d/2}} \sum_x J(x/N) \Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N] \Big)$$

for a test function J.

As before,

$$Y_t^N(J) = Y_0^N(J) + \int_0^t \left(\partial_t + N^\theta L \right) Y_s^N(J) ds + M_t^Y.$$

Compute

$$N^{\theta} L Y_t^N(J)\rangle \;=\; \frac{N^{\theta}}{N^{d/2}} \sum_x J(x/N) L\eta_t^N(x).$$

–Also, the time-derivative, using $\partial_t P_t^N = N^{\theta} P_t^N L$:

$$\partial_t Y_t^N(J)\rangle \;=\; \frac{-N^{\theta}}{N^{d/2}} \sum_x J(x/N) E_{\mu^N}[L\eta_t^N(x)].$$

–In $d = 1$,

$$L\eta_t^N(x) = \frac{1}{2}\Big\{\eta_t^N(x+1) - 2\eta_t^N(x) + \eta_t^N(x-1)\Big).$$

If we add these together,
after summing-by-parts,
we obtain

$$
\begin{aligned}
&\big(\partial_t + N^\theta L\big) Y_t^N(J) \\
&\quad = \frac{p(e)}{N^{d/2}} \sum_x \Delta J(x/N)\Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N(x)]\Big) + o(1) \\
&\quad = Y_t^N(\Delta J)\rangle + o(1).
\end{aligned}
$$

However, in the fluctuation field scaling,

the martingale

$$M_t^N = Y_t^N(J) - Y_0^N(J) - \int_0^t \left(\partial_t + N^\theta L\right) Y_s^N(J) ds$$

does not vanish.

Consider the ‘square’ martingale

$$(M_t^Y)^2 - N^\theta \int_0^t L\big(Y_s^N(J)\big)^2 - 2Y_s^N(J)LY_s^N(J)ds.$$

–Write

$$L\big(Y_s^N(J)\big)^2 = L\left(\frac{1}{N^{d/2}}\sum_x J(x/N)\Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N]\Big)\right)^2.$$

The difference $\eta(x)(1-\eta(x+1))\big(f(\eta^{xx+1})-f(\eta)\big)$ equals

$$\eta(x)(1-\eta(x+1)) \times \Bigg[\Bigg(\frac{1}{N^{d/2}}\big(J((x+1)/N)-J(x/N)\big) + \frac{1}{N^{d/2}}\sum_x J(x/N)\Big(\eta_t^N(x)-E_{\mu^N}[\eta_t^N]\Big)\Bigg)^2 - \Bigg(\frac{1}{N^{d/2}}\sum_x J(x/N)\Big(\eta_t^N(x)-E_{\mu^N}[\eta_t^N]\Big)\Bigg)^2\Bigg].$$

Then, the time integral can be computed as

$$\int_0^t \frac{p(e)}{N^d} \frac{N^\theta}{N^2} \sum_x \sum_{i=1}^d \sum_\pm \eta_s^N(x)(1 - \eta_s^N(x \pm e_i)) \times \left(\partial_{x_i} J(x/N)\right)^2 ds + o(1).$$

From the ‘GPV replacements’ already proved, this integral converges to

$$2p(e)\int_0^t \int \rho(s,u)\big(1-\rho(s,u)\big)\big|\nabla J\big|^2(u)du,$$

the quadratic variation of the martingale M_t^N limit.

With these calculations, we may formulate steps:

Step 1:

Show tightness of

$$\{Y_t^N : t \in [0, T]\} \text{ in } D([0, T], \mathcal{S}_d'),$$

where $\mathcal{S}_d'$ is the space of tempered distributions.

Initially, at $t = 0$,

$$Y_0^N(J) = \frac{1}{\sqrt{N}} \sum_x J(x/N)\big(\eta_0(x) - \rho_0(x/N)\big)$$
$$\Rightarrow Y_0(J),$$

where the mean-zero Gaussian 'white noise' field Y_0 has covariance

$$E\big[Y_0(J)Y_0(K)\big] = \int J(u)K(u)\rho_0(u)(1 - \rho_0(u))du.$$

Step 2:

Identify limit points in terms of a unique ‘infinite dimensional BM’ process.

–If Y_t is a limit point, informally

$$dY_t = p(e)\Delta Y_t + \sqrt{2p(e)\rho(t,u)(1-\rho(t,u))}\nabla \dot{W}_t$$

–Here, for $s \le t$,
if we had started from an invariant measure μ_ρ
(so $\rho(t,u) \equiv \rho$), then

$$E\big[Y_t(J)\ Y_s(K)\big] = \rho(1-\rho)\int JT_{t-s}K$$

$$E\big[\nabla \dot{W}_t(J)\ \nabla \dot{W}_s(K)\big] = \min\{s,t\}\int \nabla J \cdot \nabla K du$$

Martingale problem

More carefully, we conclude that

Y_t satisfies the martingale problem:

There is a unique distribution Q governing Y_t, concentrated on $C([0,T],\mathcal{S}'_d)$, such that

$$M_t(J) = Y_t(J) - Y_0(J) - p(e)\int_0^t Y_s(\Delta J)ds$$

$$N_t(J) = (M_t(J))^2 - 2p(e)\int_0^t\int \rho(s,u)(1-\rho(s,u))\left|\nabla J\right|^2 duds$$

are martingales.

–goes back to Holley-Stroock '79

Connection to occupation times

To give an application,
we sketch an argument for the $\mathrm{fBM}_{3/4}$ limit

$$\frac{1}{N^{3/2}} \int_0^{Nt} \eta_s(0) - 1/2 ds \Rightarrow \mathrm{fBM}_{3/4}(t).$$

One may approximate in $d = 1$ that

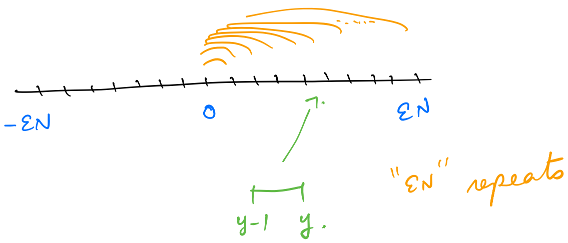
$$\lim_{\epsilon \to 0} \lim_{N \to \infty} E_{\mu_\rho} \Big| \frac{1}{N^{3/2}} \int_0^{N^2 t} \Big\{ \big(\eta_s(0) - \rho\big) - \frac{1}{N} \sum_x i_\epsilon(x/N) \big(\eta_{N^2 s}(x) - \rho\big) \Big\} ds \Big|^2 = 0$$

Indeed, the integral can be written as

$$\frac{1}{N^{3/2}} \int_0^{N^2 t} \frac{1}{2N\epsilon} \sum_{-\epsilon N}^{\epsilon N} \big(\eta(0) - \eta(x)\big) ds.$$

The sum

$$\begin{aligned} \frac{1}{N} \sum_{-\epsilon N}^{\epsilon N} \big(\eta(0) - \eta(x)\big) &= \frac{1}{N} \sum_{x=-\epsilon N}^{\epsilon N} \sum_{y=1}^{x} \big(\eta(y-1) - \eta(y)\big) \\ &\sim \frac{\epsilon N}{N} \sum_{|y| \le \epsilon N} \big(\eta(y-1) - \eta(y)\big). \end{aligned}$$


$-\varepsilon N$
0
εN
$y-1$ y.
"εN" repeats

Note the H_{-1} estimate done before

$$E_{\mu_\rho}\Big|\frac{1}{N}\int_0^{N^2 t}\big(\eta_s(y-1)-\eta_s(y)\big)ds\Big|^2 \leq Ct,$$

and $(a_1 + \cdots a_k)^2 \leq k\sum a_i^2$.

We have, in Order, that the expected difference is bounded by

$$\frac{1}{N^3}\frac{(\epsilon N)^2}{N^2}(N^2 Ct)(\epsilon N)\frac{1}{\epsilon^2} \sim \epsilon.$$

Then, the occupation time

$$\frac{1}{(N^2)^{3/4}} \int_0^{N^2 t} (\eta_s(0) - \rho) ds \sim \int_0^t \frac{N^2}{N^{3/2} N} \sum_x i_\epsilon(x/N) \big(\eta_{N^2 s}(x) - \rho\big) ds$$
$$= \int_0^t Y_s^N(i_\epsilon) ds.$$

where we recall the fluctuation field

$$Y_t^N\big(i_\epsilon\big) = \frac{1}{\sqrt{N}} \sum_x i_\epsilon(x/N) \big(\eta_{N^2 t}(x) - \rho\big).$$

Since, in the N-limit,

$$\lim_N Y_t^N\big(i_\epsilon(\cdot)\big) = Y_t(i_\epsilon(\cdot)) \stackrel{d}{=} \frac{1}{\sqrt{c}} Y_{c^2 t}\big(i_\epsilon(\cdot/c)\big)$$

we conclude

$$Z_t^\epsilon := \int_0^t Y_s(i_\epsilon) ds$$

satisfies in the ϵ-limit

$$\lim_\epsilon Z_t^\epsilon = Z_t \stackrel{d}{=} \frac{1}{c^{3/4}} Z_{ct}.$$

But, as Y_s^N limits to a Gaussian field,

conclude the N, ϵ-limit Z_t

is a continuous Gaussian process with stationary increments.

Then, Z_t is a $fBM_{3/4}(t)$.

–See Goncalves-Jara 2013; see also SS-Xu '95, SS 2000

Fluctuations in Zero-range models

In other models, without the ‘closing’ properties of symmetric exclusion, some type of ‘replacement’ will be needed.

–The theory is robust, if we start in an invariant measure μ_ρ.

–We will also discuss ‘nonequilibrium fluctuations’ for which there are more recent developments, and open questions.

Consider nearest-neighbor symmetric Zero-range processes, with initial distribution μ_ρ.

As before

$$Y_t^N(J) = \frac{1}{N^{d/2}} \sum_x J(x/N)\big(\eta_t^N(x) - \rho\big)$$

satisfies a discrete evolution equation.

–Here,

$$N^\theta L Y_t^N(J) = \frac{p(e)}{N^{d/2}} \sum_x \Delta J(x/N) g(\eta_t^N(x)) + o(1).$$

Before, in the hydrodynamic scaling, $g(\eta_t^N(x))$ could be replaced by a homogenized function of the empirical measure, namely $\phi(\eta_t^{N\epsilon}(x))$.

–The difficulty now is that, in dividing by $N^{d/2}$, we have less room to manuever, and have to include more terms in the approximation.

Boltzmann-Gibbs replacement

Let $h(\eta)$ be a local function.
Define $\hat{h}(\rho) = E_{\mu_\rho}[h]$.

Goal: In stationary or nonstationary settings,
we would like to have

$$\lim_{N\uparrow\infty} \mathbb{E}^N \Big| \int_0^t \frac{1}{N^{d/2}} \sum_x J(\tfrac{x}{N})$$

$$\times \Big\{ \tau_x h(\eta_s^N) - \hat{h}(\rho(s, \tfrac{x}{N})) - \hat{h}'(\rho(s, \tfrac{x}{N}))\big(\eta_s^N(x) - \rho(s, \tfrac{x}{N})\big) \Big\} ds \Big|^2 = 0.$$

–Recall, that $\rho(t, u) \equiv \rho$, if we start from the 'equilibrium' μ_ρ.
goes back to Brox-Rost '84

In the Zero-range context,
let

$$\tau_x h(\eta) = g(\eta(x)) \text{ and } \hat{h}(\rho) = \phi(\rho).$$

Note $g(0) = 0$ and

$$\begin{aligned} E_{m_\rho}[g] &= \frac{1}{Z} \sum_{k \geq 0} \frac{g(k)\phi(\rho)^k}{g(k)g(k-1)\cdots g(1)} \\ &= \frac{\phi(\rho)}{Z} \sum_{k \geq 1} \frac{\phi(\rho)^{k-1}}{g(k-1)\cdots g(1)} \\ &= \phi(\rho). \end{aligned}$$

From the BG replacement, we have

$$\begin{aligned} g(\eta_t^N(x)) &\sim E\Big[g(\eta_t^N(x))|\eta_{N^\theta t}^{(N\epsilon)}\Big] \\ &\sim \phi(\rho(t,x/N)) + \phi'(\rho(t,x/N))\Big(\eta_{N^\theta t}^{(N\epsilon)} - \rho(t,x/N)\Big). \end{aligned}$$

In 'equilibrium' starting under μ_ρ, the associated martingale $M_t^N(J)$ has expected quadratic variation,

$$\int_0^t \frac{2p(e)}{N^d} \sum_x \left|\nabla J\right|^2 (x/N) g(\eta_s^N(x)) ds$$
$$\sim 2p(e)\phi(\rho)\|\nabla J\|_{L^2}^2 t.$$

With the ‘equilibrium’ or stationary BG principle,
we obtain

$$dY_t(H) = p(e)\phi'(\rho)Y_t(\Delta H) + \sqrt{2p(e)\phi(\rho)}\nabla \dot{W}_t,$$

more precisely written in the martingale problem format.

–Here, $\nabla \dot{W}_t$ is as before.

Comments

1. The Boltzmann-Gibbs estimate, under ‘equilibrium’ μ_ρ, was proved in Brox-Rost. This proof works for a variety of processes. Other proofs are also available.

2. There are only few results, with respect to Boltzmann-Gibbs, starting out of the invariant measure, in more general symmetric systems.

See Chang-Yau ’92, Jara-Menezes 2018,

and Landim-SS 2025+ (in progress).

3. But, for asymmetric systems, in $d=1$, there is now a wealth of results, and different methods to derive KPZ equations and KPZ class limits, with scalings:

Space $1/N$, time $N^{3/2}$, deviation $1/\sqrt{N}$.

See review Corwin-Shen 2020, Huang-Matetski-Weber 2025

4. For asymmetric systems, in $d \geq 2$, there are less results.

Chang-Landim-Olla 2001, Landim-Olla-Varadhan 204, Caravenna-Sun-Zygouras '20, Chatterjee-Dunlap '20, Gu '21, Cannizzaro-Erhard-Toninelli '21

Some ideas for the BG replacement

Recall that we would like to show that the L^1 norm of

$$\int_0^t \frac{1}{N^{d/2}} \sum J(x/N)\Big(\tau_x h(\eta_s^N) - \hat{h}(\rho) - \tilde{\hat{h}}'(\rho)\big(\eta_s^N(x) - \rho\big)\Big] ds$$

vanishes in the $N \uparrow \infty$ limit.

We can write it as the sum of two terms:

$$\int_0^t \sum_x N^{-d/2} J(x/N)\Big[\tau_x h(\eta_s^N) - E[\tau_x h(\eta_s^N)| \sum_{y\in B_{\ell,x}} \eta_s^N(y)]\Big] ds$$
$$+ \int_0^t \sum_x N^{-d/2} J(x/N)\Big[E[\tau_x h(\eta_s^N)| \sum_{y\in B_{\ell,x}} \eta_s^N(y)]$$
$$-\hat{h}(\rho) - \hat{h}'(\rho))\eta_s^N(x) - \rho)\Big] ds \;=\; A_1 + A_2.$$

–Here, $B_{\ell,x}$ is a block of width ℓ around each x where $\ell << N$ is another scaling parameter.

A strategy is to try to bound A_1 in terms of the 'local density' around x, and to use 'local CLT' type Taylor expansions with A_2.

–When in 'equilibrium', starting under μ_ρ, as mentioned there are a few options. Tying into previous development, one way is to bound the variance or H_{-1} norm of A_1; indeed, see Notes.

–Out of 'equilibrium', there are less tools. The idea in Chang-Yau is to use relative entropy bounds.

From the variational form of relative entropy:

$$H(\kappa;\nu) = \sup_G \left\{ E_\kappa[G] - \log E_\nu[e^G] \right\}.$$

Of course, when κ is absolutely continuous with respect to ν, then

$$H(\kappa;\nu) = \int \frac{d\kappa}{d\nu} \log \frac{d\kappa}{d\nu} d\nu.$$

One has

$$E_\kappa[G] \leq \frac{1}{\gamma} H(\kappa;\nu) + \frac{1}{\gamma} \log E_\nu[e^{\gamma G}].$$

When we start out of ‘equilibrium’ in a measure κ on $\mathbb{T}_N^d$, then the relative entropy with μ_ρ is

$$H(\kappa; \mu_\rho) = O(N^d).$$

So, to bound A_1, using the relative entropy inequality, we should take $\gamma = cN^d$.

–But, then, we have to bound a term like

$$\frac{1}{cN^d} \log \mathbb{E}_{\mu_\rho}\Big[\exp\Big\{cN^{d/2}\int_0^t \sum_x J(x/N) \\ \times \Big[\tau h(\eta_s^N) - E[\tau_x h(\eta_s^N)| \sum_{y\in B_{\ell,x}} \eta_s^N(y)]\Big] ds\Big\}\Big].$$

The time average helps:
a 'Feynman-Kac' estimate can be used.

–Let us call $\mathcal{V}$ the expression in the time integral.

–For the operator $N^2 L + \mathcal{V}$:

$$\mathbb{E}_{\mu_\rho}\Big|\int_0^t \mathcal{V} ds\Big| \le \frac{1}{\gamma}\log \mathbb{E}_{\mu_\rho}\Big[e^{|\int_0^t \mathcal{V} ds|}\Big]$$
$$\le 2\max_{\pm} t \sup_{f:\|f\|_{L^2}=1}\Big\{\langle \pm\mathcal{V}, f^2\rangle_{\mu_\rho} - \frac{N^2}{\gamma}D(f)\Big\}.$$

The term $\langle \pm\mathcal{V}, f^2\rangle_{\mu_\rho}$ is further estimated in terms of $D(f)$ using relative entropy and 'log-Sobolev' ineqaulities.

In Chang-Yau, dimension is limited to $d = 1$.

–The ‘log Sobolev’ inequality for the system is

$$H\Big(\frac{d\kappa}{d\mu}\Big) \leq CN^2 D\Big(\frac{d\kappa}{d\mu}\Big).$$

More recent ideas

A recent idea is to start from a 'local equilibrium measure', say in Exclusion type systems,

$$\kappa_{\rho_0} = \prod_{x \in \mathbb{T}_N^d} \mathrm{Bern}(\rho_0(x/N))$$

where ρ_0 is smooth.

–A main problem is that we do not know μ_t^N the distribution of the system at time t.

–However, given the hydrodynamic density $\rho(t,u)$, it might not be far from

$$\nu_t = \prod_{x \in \mathbb{T}_N^d} \mathrm{Bern}(\rho(t, x/N)).$$

In fact, Jara-Menezes 2018 show that

$$H(\mu_t^N; \nu_t) \leq C \begin{cases} 1 & d = 1 \\ \log N & d = 2 \\ N^{d-2} & d \geq 3. \end{cases}$$

–With such bounds, they are able to show for types of Exclusion models the Boltzman-Gibbs estimate for $d \leq 3$.

–An open problem is understand BG estimates for more general processes (Landim-SS 2025+ in progress) and importantly in $d \geq 4$ (new ideas are needed)!

Differentiating relative entropy

We end with a useful lemma about the growth of $H(f_t^N) = H(\mu_t^N; \nu_0)$, of its own interest.

–Here,

$$f = f_t^N = \frac{d\mu_t^N}{\nu_0}.$$

Lemma

$$\frac{d}{dt} H(f_t^N) \leq -D_N(\sqrt{f}) + \int \left(L_N^{*,\nu_0} 1\right) f d\nu_0$$

–We remark when $\nu_0 = \mu_\rho$ invariant in Exclusion say, then L_N^{*,μ_ρ} is the generator of the process with reversed rates, and $L_N^{*,\mu_\rho} 1 = 0$.

–GPV 1988, Yau 1991, Jara-Menezes 2018

Recall the forward equation

$$\frac{d}{dt} f_t^N = L_N^{*,\nu_0} f_t^N$$

Here, in terms of (abstract) rates $r(\cdot, \cdot)$ of the process,

$$Lh(\eta) = \sum_{\zeta} r(\eta, \zeta) \left(h(\zeta) - h(\eta) \right).$$

The $L^2(\nu_0)$ adjoint is

$$L^{*,\nu} g(\eta) = \sum_{\zeta} \left(r(\zeta, \eta) \frac{\nu(\zeta)}{\nu(\eta)} g(\zeta) - r(\eta, \zeta) g(\zeta) \right).$$

Then, generically,

$$\begin{aligned}
\frac{d}{dt}H(f) &= \frac{d}{dt}\int f \log f d\nu \\
&= \int \big\{ \big(L^{*,\nu} f\big)(\log f) + L^{*,\nu} f\big) d\nu \\
&= \int \big\{ L^{*,\nu} f\big) \cdot \big(\log f + 1\big) d\nu \\
&= \int f \cdot L\big(\log f + 1) d\nu \\
&= \int f(L \log f) d\nu.
\end{aligned}$$

By the inequality

$$\begin{aligned} & a(\log a - \log b) \\ & \quad \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a}) = -\big(\sqrt{b} - \sqrt{a}\big)^2 + b - a \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt} H(f) &\leq -\int \sum_{\zeta} r(\eta,\zeta)\big(\sqrt{f(\zeta)} - \sqrt{f(\eta)}\big)^2 d\nu + \int Lf d\nu \\ &= -D(\sqrt{f}) + \int f(L^{*,\nu} 1) d\nu_\rho, \end{aligned}$$

where $D(\sqrt{f})$ is the associated Dirichlet form.