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Math 122B
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Homework 1

1. Suppose x and y are positive integers such that the sum of x^2 and y is $10x$. Find the values of x and y which maximize the square-root of the sum of $2x$ and y.

Solution. We want to find x and y that maximizes

$$A = \sqrt{2x + y}.$$

We are given that $x^2 + y = 10x$, so it follows

$$y = 10x - x^2.$$

Since we know both x and y are positive integers and $y = x(10 - x)$, it must be the case that $0 < x < 10$ (or equivalently, $1 \leq x \leq 9$, since x is an integer).

Substituting the above equation for y into A and simplifying we see

$$\begin{aligned} A &= \sqrt{2x + y} \\ &= \sqrt{2x + (10x - x^2)} \\ &= \sqrt{12x - x^2} \\ &= \left(12x - x^2\right)^{1/2}. \end{aligned}$$

We now want to determine the critical points of A, so we first find A':

$$\begin{aligned} A' &= \frac{1}{2}\left(12x - x^2\right)^{-1/2}(12 - 2x) \\ &= \frac{12 - 2x}{2\sqrt{12x - x^2}} \\ &= \frac{6 - x}{\sqrt{x(12 - x)}}. \end{aligned}$$

Clearly $A' = 0$ when $x = 6$ so this is the critical point. (Note: We do not need to consider the values for which the derivative is undefined since these x-values are all outside the domain of x!)

Evaluating A at our critical point and endpoints we see:

- $x = 1 \Longrightarrow A = \left(12(1) - (1)^2\right)^{1/2} = \sqrt{11} \approx 3.317$
- $x = 6 \Longrightarrow A = \left(12(6) - (6)^2\right)^{1/2} = \sqrt{36} = 6$
- $x = 9 \Longrightarrow A = \left(12(9) - (9)^2\right)^{1/2} = \sqrt{27} \approx 5.196$

Thus, $x = 6$ is a global maximum and so $x = 6$ and $y = 10(6) - (6)^2 = 24$ are the desired integers.