415A/515A Exam 2
Practice Problems

The following problems are related to the course material found in Sections 8-11 and 13-15 in the textbook. These problems do not reflect the types (or difficulty) of problems you may see on Exam 2 (in fact, these problems were selected having no knowledge of Exam 2 at all), nor does it cover every aspect of the material from those sections; however, they should all be problems you can complete with your knowledge of the material thus far. Please let me know if there are any errors or typos, and if anything could use clarification please don't hesitate to ask. Here is my email once again for reference: `rwilliams@math.arizona.edu`.

1. Write each of the following as a single cycle or a product of disjoint cycles:

   (a.) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$ (b.) $(1\ 2)(1\ 3)(1\ 4)$

   (c.) $(1\ 3)^{-1}(2\ 4)(2\ 3\ 5)^{-1}$ (d.) $(1\ 4\ 5)(1\ 2\ 3\ 5)(1\ 3)$

2. How many elements of S_n map n to n?
3. Determine the right cosets of $\langle 3 \rangle$ in $\mathbb{Z}_{12}$.
4. Define the *diagonal subgroup* of $\mathbb{R} \times \mathbb{R}$ to be $H = \{(x, x) : x \in \mathbb{R}\}$ where the operation is addition in each component. Describe the right cosets of H in $\mathbb{R} \times \mathbb{R}$ geometrically.
5. Let A and B be groups. Prove that each right coset of $A \times \{e\}$ in $A \times B$ contains precisely one element from $\{e\} \times B$.
6. Prove or disprove: The direct product of two cyclic groups is cyclic.
7. Prove that if $a, b \in \mathbb{Z}$ then $\langle a, b \rangle = \langle d \rangle$, where $d = (a, b)$ (the greatest common divisor of a and b). Note: The notation $\langle a, b \rangle$ denotes the subgroup generated by the two elements a and b.
8. Compute $[\mathbb{Z}_{40} : \langle 12, 20 \rangle]$.
9. Assume that G is a group with a subgroup H such that $|G| < 45$, $|H| > 10$ and $[G : H] > 3$. Find $|G|, |H|$ and $[G : H]$.
10. Determine the isomorphism class of $\mathbb{Z}_{18}/\langle 3 \rangle$.
11. Let G be a simple group with $\phi : G \to G$ any homomorphism. Prove that $\phi(G) \cong \{e\}$ or $\phi(G) \cong G$ (i.e., the image of G under ϕ is either the trivial group or all of G).
12. True or False: A homomorphism is surjective if and only if its kernel equal the identity.
13. True or False: If $\phi : G \to H$ is a homomorphism, then $\ker \phi$ is a subgroup of G.
14. Let $N \lhd G$, prove that if $[G : N]$ is prime, then G/N is cyclic. Does the converse hold?
15. Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.