5b Let $(G, \cdot)$ be a cyclic group of order n. For $k \in \mathbb{N}$ define $\phi_k(a) = a^k$ for all $a \in G$. For what values of k, if any, is ϕ_k an isomorphism? Prove your result.

Claim: ϕ_k is an isomorphism if and only if $(n, k) = 1$.

Proof. Suppose $G = \langle a \rangle$, then we have the following equivalences (Note, the $\Updownarrow$ means the same as $\Leftrightarrow$, which is the notation for "if and only if"):

$$\begin{aligned} &(n,k) = 1 \\ &\Updownarrow \quad (1) \\ &\langle a \rangle = \langle a^k \rangle \\ &\Updownarrow \quad (2) \\ G = \langle a \rangle = \langle \phi_k(a) \rangle &= \phi_k(\langle a \rangle) = \phi_k(G) \\ &\Updownarrow \quad (3) \\ &\phi_k \text{ is surjective} \\ &\Updownarrow \quad (4) \\ &\phi_k \text{ is an isomorphism.} \end{aligned}$$

Now we want to justify equivalences (1)-(4).

Equivalence (1) follows from Theorem 6.14, taking $s = 1$ and $t = k$.

Equivalence (2) follows from simple substitutions: $\phi_k(a) = a^k$ and $\langle a \rangle = G$; and the fact that for a cyclic group generated by a, $\langle \phi_k(a) \rangle = \phi_k(\langle a \rangle)$ since ϕ_k is a homomorphism – as we saw in my office.

Equivalence (3) is simply by the definition of a surjective map.

Equivalence (4) follows since ϕ_k is a homomorphism between cyclic groups of the same order (implying that surjectivity is the same as injectivity, providing the isomorphism).

Thus, combining all of the "if and only if" statements it follows that $(n, k) = 1$ if and only if ϕ_k is an isomorphism, thus proving the claim.

□