

415A/515A Exam 1
Practice Problems
(Solutions)

1. Let $S = \{w, x, y, z\}$ and $T = \{1, 2, 3, 4\}$, and define $f : S \rightarrow T$ and $g : S \rightarrow T$ by $f(w) = 2$, $f(x) = 4$, $f(y) = 1$, $f(z) = 2$ and $g(w) = 4$, $g(x) = 2$, $g(y) = 3$, $g(z) = 1$.
- (a) Is f injective? Is g injective? Is f surjective? Is g surjective?
- (b) Let $A = \{w, y\}$ and $B = \{x, y, z\}$. Determine each of the following subsets of T : $f(A)$, $g(B)$, $f(A \cap B)$, $g(A \cup B)$.

Solution.

- (a) f is not injective since $f(w) = 2 = f(z)$, g is injective, f is not surjective since $3 \notin f(S)$, and g is surjective.
- (b) $f(A) = \{1, 2\}$, $g(B) = \{1, 2, 3\}$, $f(A \cap B) = \{1\}$ and $g(A \cup B) = S$.

2. For each $n \in \mathbb{Z}$ let the mapping $f_n : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f_n(x) = nx$. For which values of n is f_n surjective? Injective?

Solution. We have that f_n is surjective if and only if $n \in \{\pm 1\}$, and injective if and only if $n \neq 0$. (The proofs of these facts shouldn't be too hard to come up with.)

3. Prove that if $f : S \rightarrow T$ and A and B are subsets of S , then $f(A \cup B) = f(A) \cup f(B)$.

Proof. To show equality of these sets we will show that $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A \cup B) \supseteq f(A) \cup f(B)$.

First we will show that $f(A \cup B) \subseteq f(A) \cup f(B)$. To this end, let $x \in f(A \cup B)$ then we know that there must exist some $y \in A \cup B$ such that $f(y) = x$. Since $y \in A \cup B$ then $y \in A$ or $y \in B$ (or both). Without loss of generality, suppose $y \in A$ then we have $x = f(y) \in f(A)$ and hence $x \in f(A) \cup f(B)$ implying $f(A \cup B) \subseteq f(A) \cup f(B)$.

Next, we want to show that $f(A) \cup f(B) \subseteq f(A \cup B)$, so let $w \in f(A) \cup f(B)$, then we know either $w \in f(A)$ or $w \in f(B)$ (or both). Again, without loss of generality, suppose $w \in f(A)$ then there exists some $z \in A$ such that $f(z) = w$. Since $z \in A \subset A \cup B$ it then follows that $w = f(z) \in f(A \cup B)$ implying $f(A) \cup f(B) \subseteq f(A \cup B)$.

Since we have shown both inclusions it then follows that we have $f(A \cup B) = f(A) \cup f(B)$ as desired. □

4. Find all the solutions to the equation $z^3 + 2i = 0$.

Solution. We want to determine the z which satisfy $z^3 = -2i$; i.e., we need to compute $(-2i)^{1/3}$. We will proceed by first rewriting $-2i$ in polar form, then applying a consequence of de Moivre's theorem:

$$\begin{aligned} (-2i)^{1/3} &= \left(2 \left(\cos \left(\frac{3\pi}{2} \right) + i \sin \left(\frac{3\pi}{2} \right) \right) \right)^{1/3} \\ &= 2^{1/3} \left(\cos \left(\frac{3\pi}{6} + \frac{2k\pi}{3} \right) + i \sin \left(\frac{3\pi}{6} + \frac{2k\pi}{3} \right) \right) \quad \text{for } k = 0, 1, 2, \\ &= 2^{1/3} \left(\cos \left(\frac{\pi}{2} + \frac{2k\pi}{3} \right) + i \sin \left(\frac{\pi}{2} + \frac{2k\pi}{3} \right) \right) \quad \text{for } k = 0, 1, 2. \end{aligned}$$

Hence, the solutions to $z^3 + 2i = 0$ are given by

$$\begin{aligned} z &\in \left\{ 2^{1/3} \left(\cos \left(\frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} \right) \right), 2^{1/3} \left(\cos \left(\frac{7\pi}{6} \right) + i \sin \left(\frac{7\pi}{6} \right) \right), 2^{1/3} \left(\cos \left(\frac{11\pi}{6} \right) + i \sin \left(\frac{11\pi}{6} \right) \right) \right\} \\ &= \left\{ 2^{1/3}i, 2^{-2/3}(-\sqrt{3} - i), 2^{-2/3}(\sqrt{3} - i) \right\}. \end{aligned}$$

5. Complete the following table in such a way that $*$ is commutative, has an identity element, and each element has an inverse:

$*$	w	x	y	z
w				x
x	z	w		
y				
z				w

Solution.

$*$	w	x	y	z
w	y	z	w	x
x	z	w	x	y
y	w	x	y	z
z	x	y	z	w

6. Verify that $\{2^m 3^n : m, n \in \mathbb{Z}\} \subset \mathbb{R}$ is a group with respect to multiplication. Is it also a group with respect to addition? Why?

Solution. For notation's sake, let $G = \{2^m 3^n : m, n \in \mathbb{Z}\}$. Since G is a subset of the multiplicative group \mathbb{R} , if we can show that G is a subgroup of \mathbb{R} , then it will follow that G is itself a group.

Clearly $1 = 2^0 3^0 \in G$, and $1 \cdot 2^m 3^n = 2^m 3^n$ for any $n, m \in \mathbb{Z}$, so G contains the identity element.

If $m, n \in \mathbb{Z}$, then $-m, -n \in \mathbb{Z}$ as well (clearly, since \mathbb{Z} is a group under $+$) so that $2^{-m} 3^{-n} \in G$. We see $2^m 3^n \cdot 2^{-m} 3^{-n} = 2^m 2^{-m} 3^n 3^{-n} = 2^0 3^0 = 1$ since $G \subset \mathbb{R}$ and \mathbb{R} is commutative under \cdot , and this simplifies to $2^m 3^n \cdot 2^{-m} 3^{-n} = 2^0 3^0 = 1$. Therefore, each element of G has an inverse which is also an element of G .

Finally, if $2^{m_1} 3^{n_1}, 2^{m_2} 3^{n_2} \in G$, then

$$2^{m_1} 3^{n_1} \cdot 2^{m_2} 3^{n_2} = 2^{m_1} 2^{m_2} 3^{n_1} 3^{n_2} = 2^{m_1+m_2} 3^{n_1+n_2} \in G,$$

where the first equality follows using the commutativity of \cdot in \mathbb{R} . Thus G is closed under \cdot .

With these properties shown, it follows that $G \leq \mathbb{R}$, and hence G is a group with respect to multiplication.

The subset $\{2^m 3^n : m, n \in \mathbb{Z}\}$ of \mathbb{R} is not a group with respect to addition for many reasons. There is no additive identity ($0 \neq 2^m 3^n$ for any $n, m \in \mathbb{Z}$), it is not closed under $+$, nor do additive inverses exist.

7. Find a subgroup of \mathbb{Q} (under the operation $+$) that contains \mathbb{Z} but is different from both \mathbb{Z} and \mathbb{Q} .

Solution. There are (infinitely) many subgroups which will work; one of which is $\langle \frac{1}{2} \rangle$. This is a subgroup of \mathbb{Q} by definition, and for any $n \in \mathbb{Z}$ we have $n = 2n \left(\frac{1}{2} \right) \in \langle \frac{1}{2} \rangle$, so $\mathbb{Z} \leq \langle \frac{1}{2} \rangle$. However, $\frac{1}{3} \neq k \left(\frac{1}{2} \right)$ for any $k \in \mathbb{Z}$, so it follows $\mathbb{Q} \not\subset \langle \frac{1}{2} \rangle$.

8. Write 2 as a linear combination of 52 and 126.

Solution. Repeatedly using the Euclidean Algorithm we see

$$\begin{aligned}126 &= 52(2) + 22 \\52 &= 22(2) + 8 \\22 &= 8(2) + 6 \\8 &= 6(1) + 2 \\6 &= 2(3) + 0.\end{aligned}$$

Using this we see that 2 is the greatest common divisor of 52 and 126, and that we may write

$$\begin{aligned}2 &= 8 - 6(1) \\6 &= 22 - 8(2) \\8 &= 52 - 22(2) \\22 &= 126 - 52(2).\end{aligned}$$

Making substitutions it then follows

$$\begin{aligned}2 &= 8 - 6 \\&= 8 - (22 - 8(2)) \\&= 8(3) - 22 \\&= (52 - 22(2))(3) - 22 \\&= 52(3) - 22(7) \\&= 52(3) - (126 - 52(2))(7) \\&= 52(17) - 126(7).\end{aligned}$$

Hence, $2 = 52(17) - 126(7)$.

9. Prove that if n is odd, then $\phi(2n) = \phi(n)$.

Proof. Given that n is odd we know $(2, n) = 1$, and since ϕ is a multiplicative function it follows that $\phi(2n) = \phi(2)\phi(n) = \phi(n)$ (because $\phi(2) = 1$). □

10. Prove that if $(a, b) = 1$, $a|m$ and $b|m$, then $ab|m$.

Proof. Since $a|m$ there exists some $r \in \mathbb{Z}$ such that $ar = m$. Using this, $b|m$ is equivalent to $b|ar$. Since $(a, b) = 1$ it then follows that we must have that $b|r$. Now, $b|r$ so there exists some $s \in \mathbb{Z}$ such that $bs = r$. Thus, $ar = m$ and $bs = r$ so it follows $abs = m$ which implies $ab|m$, as desired. □

11. Let G be a group such that the order of each nonidentity element is 2. Prove that G is abelian.

Proof. Suppose that G were not abelian, i.e., that there exist $a, b \in G$ such that $ab \neq ba$. If this were the case, it would then follow that $(ab)(ab) \neq (ba)(ab) = ba^2b = b^2 = 1$ (since both a and b have order 2, $a^2 = b^2 = 1$). However, since G is a group, $ab \in G$ and therefore must have order 2 which contradicts what we have shown above. Thus G must be abelian. (Note, by assumption ab will be a nonidentity element. If $ab = 1$, then $b = a^{-1}$ but since a has order 2, $b = a^{-1} = a$, reducing our equation to $ab = aa = ba$ which does not satisfy $ab \neq ba$.) □

12. Fill in the blanks in the following table to obtain a group isomorphic to $\mathbb{Z}/4\mathbb{Z}$:

*	a	b	c	d
a				
b				
c				
d				

Solution. There are several possibilities here, we will present two:

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

*	a	b	c	d
a	d	c	a	b
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

The first group is isomorphic via the map: $a \mapsto [0], b \mapsto [1], c \mapsto [2], d \mapsto [3]$. While the second group is isomorphic using the map: $a \mapsto [3], b \mapsto [1], c \mapsto [0], d \mapsto [2]$.

13. Suppose that G and H are groups with $G \cong H$. Prove that if G has an element of order n , then H has an element of order n .

Proof. Let $\phi : G \rightarrow H$ be the isomorphism between G and H , and suppose that $g \in G$ has order n . Now $\phi(g) \in H$ and using the properties of the isomorphism, $\phi(g)^n = \phi(g^n) = \phi(1) = 1$ since isomorphisms must take identity elements to identity elements. Thus $\phi(g)$ has order n , so H has an element of order n .

□