1. (System/Aug. Mat.) Write an augmented matrix for the system

$$\begin{cases} x_1 + 2x_2 = 4 \\ -3x_1 + x_3 = -1 \\ 2x_1 + 4x_2 - x_3 = 0 \\ x_2 + x_3 = 2 \end{cases}$$

2. (Consist/Inf. Sols) Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right).$$

   (a) Is it possible for this system to be inconsistent? Explain.

   (b) For what values of β will the system have infinitely many solutions?

3. (Transpose/Matrix Equivalence) A matrix A is said to be *skew symmetric* if $A^T = -A$. Suppose that A is a 2×2 skew symmetric matrix, determine the diagonal entries of A.

4. (Matrix Inverse) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}.$$

5. (Span) Let

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}.$$

   Determine whether $\mathbf{x} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

6. (Basis/Dimension) Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix}$$

   where a, b, c are real numbers. What is the dimension of S?

7. (Basis/Linear Independence) The vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   span $\mathbb{R}^3$. Reduce the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

8. (Coordinate Vectors/Spanning Set) Determine if the set $\{x-2, 2x-3, x^2-1\}$ is a spanning set for $\mathbb{P}_2$.

9. (Linear Transform./Kernel/Image) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - 2x_3 \\ 3x_1 + 2x_2 + x_3 \end{pmatrix}.$$

(a) Find the matrix A associated to T.

(b) Determine $\ker L$.

10. (Subspaces) Determine whether the following sets are subspaces of the corresponding vector spaces:

    (a) The set of all 2×2 matrices A in $M_{2\times 2}(\mathbb{R})$ such that $\det A = 0$.

    (b) The set of all polynomials $p(x)$ in $\mathbb{P}_4$ such that $p(0) = 0$.

    (c) The set of all nondecreasing functions in $C[-1, 1]$.

    (d) Fix a 2×2 matrix A. The set of all 2×2 matrices B in $M_{2\times 2}(\mathbb{R})$ such that $AB \neq BA$.

11. (Column/Row/Null Space) Given the matrix

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix},$$

    determine a basis for the following subspaces:

    (a) $\text{Col}(A)$

    (b) $\text{Row}(A)$

    (c) $\text{Nul}(A)$

12. (Null Space) Let $A = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$ be a 4×3 matrix and suppose that the vectors

$$\mathbf{z}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{z}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

    form a basis for $\text{Nul}(A)$. If $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, find all solutions of the system $A\mathbf{x} = \mathbf{b}$.

13. (Characteristic Equation) Determine the characteristic equation for the matrix

$$\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}.$$

    Simplify, but do not factor!

14. (Eigenvalues) Let A be a 2×2 matrix. If $\text{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

15. (Projection - Line) Find the point on the line $y = 2x + 1$ that is closest to the point $(5, 2)$.

16. (Orthogonality) Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be vectors in $\mathbb{R}^3$ such that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$ and $\mathbf{x}_2 \cdot \mathbf{x}_3 = 0$. Is it true that $\mathbf{x}_1 \cdot \mathbf{x}_3 = 0$? Prove or provide a counterexample.

17. (Orthogonal Complement) Let S be the subspace of $\mathbb{R}^3$ spanned by

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

    Find a basis for $S^\perp$.

18. (Orthonormal Basis) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for a vector space V, and let

$$\mathbf{a} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3, \qquad \mathbf{b} = \mathbf{u}_1 + 7\mathbf{u}_3.$$

    Determine the following

(a) $\mathbf{a} \cdot \mathbf{b}$

(b) $\|\mathbf{b}\|$

19. (Least Squares) Find a least squares solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}.$$

20. (T/F - Orthogonal Complement) It is possible for a matrix to have the vector $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ in its row space and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ in its null space.

21. (T/F - Rank/Inv. Mat. Thm.) Let A be an $m \times n$ matrix with $\text{Rank}(A) = n$. If $A\mathbf{c} = A\mathbf{d}$ then we must have $\mathbf{c} = \mathbf{d}$.

22. (T/F - Orthonormal Basis) The set

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

forms an orthonormal basis of $\mathbb{R}^2$.

23. (T/F - Eigenvalue) Let A be an invertible matrix and suppose λ is an eigenvalue of A, then $1/\lambda$ must be an eigenvalue of A^{-1}.

24. (T/F - Eigenvectors) Let A be an $n \times n$ matrix and let λ be a nonzero eigenvalue of A. If $\mathbf{x}$ is an eigenvector corresponding to λ, then $\mathbf{x}$ must be in the row space of A.