The following review problems may help you prepare for Exam 2.

True/False Questions:

1. Let U be a subspace of a vector space V. If $\mathbf{u}$ and $\mathbf{v}$ are elements of V but not elements of U, it is still possible for $\mathbf{u}+\mathbf{v}$ to be an element of U.

2. If A is a 4×6 matrix, the null space of A can never be $\{\mathbf{0}\}$.

3. If A is a 4×4 matrix with real-valued entries whose null space is $\{\mathbf{0}\}$, its column space must be all of $\mathbb{R}^4$.

4. If A and B are any two row equivalent matrices, then A and B must have the same null space.

5. Every collection of four nonzero vectors in a five dimensional vector space is linearly independent.

Properties of Matrices:

1. If $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, describe the effect multiplying a matrix A on the left by E^5 would have, i.e., describe $E^5 A$.

2. Let $A = \begin{pmatrix} 2 & 5 \\ 6 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 0 & -6 \end{pmatrix}$. Find an elementary matrix E such that $EA = B$.

3. Define the following matrices

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

   (a) Compute $\det A$.
   (b) Compute A^{-1}.
   (c) Compute $(B^T)^{-1}$.

4. Suppose A is a 3×5 matrix and B is a 5×3 matrix. Is it possible or impossible for AB to ever equal BA?

5. Solve $A\mathbf{x} = \mathbf{b}$, where

$$A^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

6. Find the inverse of the matrix
$$\begin{pmatrix} 1 & 0 & k \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$
where k is any real number.

Null Space / Column Space:

1. If A is an $n \times n$ invertible matrix, what is the null space of A? The column space?

2. Suppose A is a matrix such that,

   - a basis for Null(A) is $\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}$, and
   - a basis for Col(A) is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

   Give an example of one such matrix A.

Vector Spaces:

1. The following sets are not vector spaces. Give one reason why they are not.

   (a) The set of polynomials of degree 2.

   (b) The set of all positive real numbers.

Subspaces:

1. Consider the following subset of $\mathbb{R}^3$:
$$H = \left\{ \begin{pmatrix} a \\ a+1 \\ b \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$
   Is H a subspace of $\mathbb{R}^3$? Explain.

Bases:

1. Determine $[\mathbf{u}]_{\mathcal{B}}$ given $\mathbf{u} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$ and the basis $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

2. Consider the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2, determine whether the set $\{1 + x, x - x^2\}$ is a basis for $\mathbb{P}_2$.

3. Consider the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2, determine whether the set $\{1+x, 1-x, x+x^2, x-x^2\}$ is a basis for $\mathbb{P}_2$.

4. The following is a basis for $M_{2\times 2}$:

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

   (a) Find the coordinate vector for $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ relative to B.

   (b) Determine if the matrices

$$\left\{ \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} \right\}$$

   are linearly independent using coordinate vectors.

   (c) The following is another basis for $M_{2\times 2}$:

$$C = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   Find the change of coordinate matrix from C to B.

5. The set $S = \{3 \times 3 \text{ symmetric matrices}\}$ is a subspace of $M_{3\times 3}$. Find a basis for S. What is the dimension of the subspace S?

Linear Independence:

1. Suppose that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in a vector space V. Let c be a nonzero scalar. Must the following sets also be linearly dependent? (Yes/No)

   (a) $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3\}$

   (b) $\{\mathbf{v}_1, c\mathbf{v}_2, \mathbf{v}_3\}$

   (c) $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{0}\}$