True/False Questions:

1. If the row echelon form of A involves free variables, then the system $A\mathbf{x} = \mathbf{b}$ will always have infinitely many solutions.

2. Let $A = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$ be a 4×3 matrix with $\mathbf{a}_2 = \mathbf{a}_3$. If $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$ then the system $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.

3. A homogeneous linear system is always consistent.

4. If $\text{span}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \text{span}(\mathbf{a}, \mathbf{b})$ then the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly dependent.

REF/RREF:

1. Are the following matrices in REF, RREF, or neither?

   (a) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 4 & -8 & 4 & 20 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 5 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 7 \\ 3 \\ 5 \\ 9 \\ 2 \end{pmatrix}$

Matrix Equation:

1. Suppose A and $\mathbf{b}$ be given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

   Find $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ and write your answer in parametric vector form.

Linear Independence:

1. Determine if the following sets of vectors are linearly independent in $\mathbb{R}^3$:

   (a) $\{\mathbf{0}\}$

   (b) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{0}\}$

   (c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

   (d) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

2. Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be distinct vectors in $\mathbb{R}^n$ such that $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \mathbb{R}^n$. Is the set $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ linearly independent? Why or why not?

Linear Transformation:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a map where T is given by:

$$T(\mathbf{x}) = T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix}.$$

   (a) Is T a linear transformation?

   (b) Is T one-to-one?

   (c) Is T onto?

2. Determine the standard matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{pmatrix} 2x_3 \\ x_2 + 3x_1 \\ 2x_1 - x_3 \end{pmatrix}.$$

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation.

   (a) If $T(\mathbf{x}) = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$ is T one-to-one? Onto?

   (b) If $T(\mathbf{x}) = \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix}$ is T one-to-one? Onto?

   (c) If $T(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}$ is T one-to-one? Onto?

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation.

   (a) Is it possible for T to be onto? If so, give an example.

   (b) Is it possible for T to be one-to-one? If so, give an example.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

   (a) Is it possible for T to be onto? If so, give an example.

   (b) Is it possible for T to be one-to-one? If so, give an example.

Equivalent Statements:

1. Determine whether the following are equivalent statements:

   (a) The columns of A span $\mathbb{R}^n \overset{?}{\Leftrightarrow}$ The columns of A are linearly independent.

   (b) For every $\mathbf{b} \in \mathbb{R}^m$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution $\overset{?}{\Leftrightarrow}$ The matrix A has no free variables.

   (c) $\mathbf{b}$ is a linear combination of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \overset{?}{\Leftrightarrow}$ The augmented matrix $(\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n \mid \mathbf{b})$ is consistent.

   (d) The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent $\overset{?}{\Leftrightarrow}$ Every vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be written in terms of the other vectors.