Homework 6 Solutions

Sections 6.1 - 6.4

Section 6.1

28. Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$.

Solution. As per the hint, an arbitrary $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$ is of the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$, so we see that

$$\mathbf{y} \cdot \mathbf{w} = \mathbf{y} \cdot (c_1\mathbf{u} + c_2\mathbf{v}) = c_1\mathbf{y} \cdot \mathbf{u} + c_2\mathbf{y} \cdot \mathbf{v} = c_1(0) + c_2(0) = 0,$$

where the second-to-last equality follows since $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y}$.

Section 6.2

10. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s, where

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Solution. A simple computation shows that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $1 \leq i, j \leq 3$ with $i \neq j$, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal *set* in $\mathbb{R}$. Since (nonzero) orthogonal vectors are linearly independent, it follows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set and therefore forms a basis of $\mathbb{R}^3$ (since there are 3 vectors and the dimension of $\mathbb{R}^3$ is 3).

Now we have

$$\begin{aligned}
\mathbf{x} &= \frac{\mathbf{u}_1 \cdot \mathbf{x}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{x}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{x}}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 \\
&= \frac{15+9+0}{9+9+0}\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \frac{10-6-1}{4+4+1}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \frac{5-3+4}{1+1+16}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \\
&= \frac{4}{3}\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.
\end{aligned}$$

16. Let $\mathbf{y} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

Soluion. If we let L denote the line through $\mathbf{u}$ and the origin, then the distance from $\mathbf{y}$ to L is given by $\|\mathbf{y} - \text{proj}_L\mathbf{y}\|$. Thus, we begin by computing the projection:

$$\text{proj}_L\mathbf{y} = \frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{-3+18}{1+4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Hence we have

$$\mathbf{y} - \text{proj}_L\mathbf{y} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix},$$

and so

$$\|\mathbf{y} - \text{proj}_L \mathbf{y}\| = \left\| \begin{pmatrix} -6 \\ 3 \end{pmatrix} \right\| = \sqrt{(-6)^2 + 3^2} = 3\sqrt{5}.$$

Section 6.3

10. Let W be the subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W where

$$\mathbf{y} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Solution. We seek to write $\mathbf{y}$ as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\mathbf{y} \in W$ and $\mathbf{z} \in W^{\perp}$, but we know these vectors will automatically be given by $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Computing these vectors,

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{y}}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 \\
&= \frac{3+4-6}{1+1+1}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \frac{3+5+6}{1+1+1}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \frac{-4+5-6}{1+1+1}\begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \frac{14}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \frac{-5}{3}\begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix}.
\end{aligned}$$

and

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix},$$

we have

$$\mathbf{y} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

14. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ where

$$\mathbf{z} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

Solution. The set of vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is simply the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e., $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, so to find the best approximation to $\mathbf{z}$ by vectors in this span we just need to project $\mathbf{z}$ onto it. For notation, let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then the best approximation to $\mathbf{z}$ will be given by

$$\text{proj}_W \mathbf{z} = \frac{\mathbf{v}_1 \cdot \mathbf{z}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{z}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{4+3}{4+1+9}\mathbf{v}_1 + \frac{10-8-2}{25+4+16+2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1.$$

Section 6.4

12. Find an orthogonal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{pmatrix}.$$

Solution. We proceed using the Gram-Schmidt Process:

$$\begin{aligned}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and we let } L_1 = \text{Span}\{\mathbf{v}_1\}, \\
\mathbf{v}_2 &= \begin{pmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{pmatrix} - \text{proj}_{L_1} \begin{pmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{pmatrix} - \frac{3+3+5+5}{1+1+1+1} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \\ 0 \\ 4 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and we let } L_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\},
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathbf{v}_3 &= \begin{pmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{pmatrix} - \mathrm{proj}_{L_2} \begin{pmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{pmatrix} - \frac{5-1+2+8}{1+1+1+1} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{-5+1+6+2+8}{1+1+4+1+1} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{pmatrix} - \frac{7}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{pmatrix}.
\end{aligned}$$

Hence, our orthogonal basis for the column space of the given matrix is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{pmatrix} \right\}.$$