Homework 5 Solutions

Sections 5.1 - 5.3

<u>Section 5.1</u>

6. Is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$? If so, find the eigenvalue.

   Solution. Note that

$$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

   so yes, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = -2$.

16. Find a basis for the eigenspace corresponding to the given eigenvalue:

$$\begin{pmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad \lambda = 4.$$

   Solution. First we note that

$$\left(\begin{array}{cccc|c} 3-4 & 0 & 2 & 0 & 0 \\ 1 & 3-4 & 1 & 0 & 0 \\ 0 & 1 & 1-4 & 0 & 0 \\ 0 & 0 & 0 & 4-4 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \cdots \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

   Hence, the solution in parametric form will be given by

$$\mathbf{x} = \begin{cases} x_1 & = & 2x_3 \\ x_2 & = & 3x_3 \\ x_3 & \text{is} & \text{free} \\ x_4 & \text{is} & \text{free} \end{cases},$$

   which has a corresponding parametric vector form:

$$\mathbf{x} = x_3\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

   Thus, a basis for the 4-eigenspace is

$$\left\{\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Section 5.2

14. Find the characteristic polynomial of the following matrix, using either a cofactor expansion or the special formula for 3×3 determinants:

$$\begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}.$$

Solution. Using cofactor expansion along the second row we see

$$\begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = (1-\lambda)\left((5-\lambda)(-2-\lambda)-18\right),$$

which simplifies to $-\lambda^3 + 4\lambda^2 + 25\lambda - 28$.

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$\begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution. We see that

$$\left(\begin{array}{cccc|c} 5-5 & -2 & 6 & -1 & 0 \\ 0 & 3-5 & h & 0 & 0 \\ 0 & 0 & 5-5 & 4 & 0 \\ 0 & 0 & 0 & 1-5 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right) \sim \cdots \sim \left(\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & 0 & h-6 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Clearly the row echelon form will have two free variables if and only if $h = 6$, meaning the dimension of the eigenspace corresponding to $\lambda = 5$ will be two if and only if $h = 6$ (since each free variable will correspond to an eigenvector).

Section 5.3

14. Diagonalize the following matrix, if possible.

$$\begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}.$$

The eigenvalues of this matrix are: $\lambda = 5, 4$.

Solution. First, we will determine the eigenspace corresponding to the eigenvalue $\lambda = 5$ (since this eigenvalue has multiplicity 2, if the eigenspace is not 2-dimensional we will know that the matrix is not diagonalizable!). We have:

$$\left(\begin{array}{ccc|c} 4-5 & 0 & -2 & 0 \\ 2 & 5-5 & 4 & 0 \\ 0 & 0 & 5-5 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus we see that x_2 and x_3 are free variables which will give us the following basis for the 5-eigenspace:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Now, for the eigenspace corresponding to $\lambda = 4$,

$$\left(\begin{array}{ccc|c} 4-4 & 0 & -2 & 0 \\ 2 & 5-4 & 4 & 0 \\ 0 & 0 & 5-4 & 0 \end{array} \right) = \left(\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \sim \cdots \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Here x_2 is our only free variable which gives us a basis for the 4-eigenspace:

$$\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Therefore, we can write $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 0 & -2 & -\frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

Solution. Simply create a 2×2 matrix with two distinct eigenvalues (which ensures the matrix is diagonalizable) where one of the eigenvalues is zero (which ensures the matrix is not invertible). Some simple examples would be

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix},$$

where a and b are any nonzero real numbers.