

Homework 4 Solutions

Sections 3.1 & 3.2

1. Let A be the 5×5 matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 & 3 \\ 2 & 1 & 0 & 4 & -1 \\ 1 & -2 & 1 & 2 & -1 \\ -3 & 0 & 2 & 4 & 2 \\ 2 & -2 & 3 & 4 & 2 \end{pmatrix}.$$

Use the method of cofactor expansion to show that $\det A = -45$.

Solution. We begin by noting that cofactor expansion can be used on any row or column of this matrix or any of its submatrices. While your choice of row/column may not be the same as the one presented here, the process should be similar.

We will expand about column 3 (since it contains a 0).

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & -1 & 2 & -1 & 3 \\ 2 & 1 & 0 & 4 & -1 \\ 1 & -2 & 1 & 2 & -1 \\ -3 & 0 & 2 & 4 & 2 \\ 2 & -2 & 3 & 4 & 2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ -3 & 0 & 4 & 2 \\ 2 & -2 & 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 1 & -2 & 2 & -1 \\ -3 & 0 & 4 & 2 \\ 2 & -2 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ -3 & 0 & 4 & 2 \\ 2 & -2 & 4 & 2 \end{vmatrix} \\ &\quad - 2 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ 2 & -2 & 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ -3 & 0 & 4 & 2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ -3 & 0 & 4 & 2 \\ 2 & -2 & 4 & 2 \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ -3 & 0 & 4 & 2 \\ 2 & -2 & 4 & 2 \end{vmatrix} \\ &\quad - 2 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ 2 & -2 & 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 1 & -2 & 2 & -1 \\ -3 & 0 & 4 & 2 \end{vmatrix} \end{aligned}$$

Now we expand each of these 4×4 determinants along column 2.

$$\begin{aligned}
&= 2 \left(-1 \begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 4 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & -1 \\ 2 & -2 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 2 \end{vmatrix} \right) \\
&+ 1 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ -3 & 4 & 2 \end{vmatrix} \right) \\
&- 2 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -1 \\ 2 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ 2 & 4 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 1 & 2 & -1 \end{vmatrix} \right) \\
&+ 3 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -3 & 4 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ -3 & 4 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 1 & 2 & -1 \end{vmatrix} \right) \\
&= 2 \left(-1 \begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 4 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 2 \\ -3 & 0 & 4 \end{vmatrix} \right) \\
&+ 1 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ -3 & 4 & 2 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ -3 & 4 & 2 \end{vmatrix} \right) \\
&- 2 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -1 \\ 2 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ 2 & 4 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 2 & 4 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ 1 & 2 & -1 \end{vmatrix} \right) \\
&+ 3 \left(-(-1) \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -3 & 4 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & 4 & -1 \\ -3 & 4 & 2 \end{vmatrix} \right)
\end{aligned}$$

Next, we expand each of these 3×3 determinants along column 1.

$$\begin{aligned}
&= 2 \left[-1 \left(1 \begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} - (-3) \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} \right) \right. \\
&\quad + (-2) \left(2 \begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} - (-3) \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} \right) \\
&\quad \left. + (-2) \left(2 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} + (-3) \begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} \right) \right] \\
&+ 1 \left[-(-1) \left(2 \begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} - (-3) \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} \right) \right. \\
&\quad + 1 \left(1 \begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} - (-3) \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} \right) \\
&\quad \left. + (-2) \left(1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} \right) \right] \\
&- 2 \left[-(-1) \left(2 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} \right) \right. \\
&\quad + 1 \left(1 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \right) \\
&\quad - (-2) \left(1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} \right) \\
&\quad \left. + (-2) \left(1 \begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} \right) \right] \\
&+ 3 \left[-(-1) \left(2 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} + (-3) \begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} \right) \right. \\
&\quad + 1 \left(1 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \right) \\
&\quad \left. - (-2) \left(1 \begin{vmatrix} 4 & -1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} \right) \right]
\end{aligned}$$

Finally, we compute all of the 2×2 determinants and simplify.

$$\begin{aligned}
&= 2[-1(1(8-8) - (-3)(4 - (-4)) + 2(4 - (-4))) \\
&\quad +(-2)(2(8-8) - (-3)(8 - (-4)) + 2(8 - (-4))) \\
&\quad +(-2)(2(4 - (-4)) - 1(8 - (-4)) + (-3)(-4 - (-2)))] \\
&+1[-(-1)(2(8-8) - (-3)(8 - (-4)) + 2(8 - (-4))) \\
&\quad +1(1(8-8) - (-3)(-2 - 12) + 2(-2 - 12)) \\
&\quad +(-2)(1(8 - (-4)) - 2(-2 - 12) + (-3)(1 - 12))] \\
&-2[-(-1)(2(4 - (-4)) - 1(8 - (-4)) + 2(-4 - (-2))) \\
&\quad +1(1(4 - (-4)) - 1(-2 - 12) + 2(1 - 6)) \\
&\quad -(-2)(1(8 - (-4)) - 2(-2 - 12) + 2(1 - 12)) \\
&\quad +(-2)(1(-4 - (-2)) - 2(1 - 6) + 1(1 - 12))] \\
&+3[-(-1)(2(4 - (-4)) - 1(8 - (-4)) + (-3)(-4 - (-2))) \\
&\quad +1(1(4 - (-4)) - 1(-2 - 12) + (-3)(1 - 6)) \\
&\quad -(-2)(1(8 - (-4)) - 2(-2 - 12) + (-3)(1 - 12))] \\
&= 2[-((0) + 3(8) + 2(8)) \\
&\quad +(-2)(2(0) + 3(12) + 2(12)) \\
&\quad -2(2(8) - (12) - 3(-2))] \\
&+ [(2(0) + 3(12) + 2(12)) \\
&\quad +((0) + 3(-14) + 2(-14)) \\
&\quad -2((12) - 2(-14) - 3(-11))] \\
&-2[(2(8) - (12) + 2(-2)) \\
&\quad +((8) - (-14) + 2(-5)) \\
&\quad +2((12) - 2(-14) + 2(-11)) \\
&\quad -2((-2) - 2(-5) + (-11))] \\
&+3[(2(8) - (12) - 3(-2)) \\
&\quad +((8) - (-14) - 3(-5)) \\
&\quad +2((12) - 2(-14) - 3(-11))] \\
&= 2[-(40) - 2(60) - 2(10)] + [60 - 70 - 2(73)] - 2[0 + 12 + 2(18) - 2(-3)] + 3[10 + 37 + 2(73)] \\
&= 2[-180] + [-156] - 2[54] + 3[193] \\
&= -360 - 156 - 108 + 579 \\
&= -45
\end{aligned}$$

2. Let A and B be 10×10 matrices. Denote the rows of A by: A_1, A_2, \dots, A_{10} ; and the rows of B by: B_1, B_2, \dots, B_{10} . Suppose the rows of matrix B are constructed as follows:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 + 2A_4 \\ B_3 &= A_3 \\ B_4 &= A_6 \\ B_5 &= 3A_8 \\ B_6 &= A_9 \\ B_7 &= A_7 - 3A_1 \\ B_8 &= 2A_5 \\ B_9 &= A_4 \\ B_{10} &= A_{10} \end{aligned}$$

If $\det A = 3$ what is $\det B^{-1}$?

Solution. First, recall that adding a multiple of one row of A to another row of A does not change the determinant, so the changes being made in B_2 and B_7 do not affect the determinant.

Note that we have swapped the position of rows A_4, A_6 , and A_9 , which would take two swaps to move them back to their original location (swap A_6 with A_4 to put A_4 back into its original location, then swap A_6 and A_9 to put them back into their original positions). Hence, the determinant of B will require multiplication by $(-1)(-1)$ from that of A (and hence there is no affect).

We also see that A_5 and A_8 have been swapped, which will require multiplication by (-1) .

Finally, A_5 is being scaled by 2 and A_8 is being scaled by 3, which will require multiplication by $(2)(3)$.

Putting all of this together we see

$$\det B = (-1)(-1)(-1)(2)(3) \det A = -6 \det A = -6(3) = -18,$$

and therefore,

$$\det B^{-1} = \frac{1}{\det B} = \frac{1}{-18} = -\frac{1}{18}.$$