Homework 3 Solutions

Sections 2.1-2.3

Section 2.1

16. Let A, B, and C be arbitrary matrices for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

    a. If A and B are 3×3 and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.

    b. The second row of AB is the second row of A multiplied on the right by B.

    c. $(AB)C = (AC)B$

    d. $(AB)^T = A^T B^T$

    e. The transpose of a sum of matrices equals the sum of their transposes.

    Solutions.

    a. False. If A and B are both 3×3, then their product must also be 3×3, but the given $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$ is only 1×3. The product should instead be $AB = [A\mathbf{b}_1\ A\mathbf{b}_2\ A\mathbf{b}_3]$ (with no pluses between the entries).

    b. True. This is by definition of the product AB.

    c. False. Matrix multiplication is not commutative in general.

    d. False. Transposing a product reverses their order, i.e., $(AB)^T = B^T A^T$.

    e. True. This is a generalization of Theorem 3(b). If the transpose of two matrices equals the sum of their transposes, then it easily follows the transpose of an arbitrary (finite) sum of matrices equals the sum of their transposes as well.

Section 2.2

18. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.
    Solution. We begin by multiplying both sides of $A = PBP^{-1}$ on the left by P^{-1} to obtain:

    $$P^{-1}A = P^{-1}PBP^{-1},$$

    and the right-side then simplifies into: $P^{-1}A = BP^{-1}$. Now, we multiply both sides of this equation on the right by P to obtain

    $$P^{-1}AP = BP^{-1}P,$$

    which simplifies into: $P^{-1}AP = B$.

32. Find the inverse of the following matrix, if it exists. Use the algorithm introduced in this section.

    $$\begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{pmatrix}$$

    Solution. We begin by augmenting the given matrix with the 3×3 identity matrix:

    $$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right).$$

Now, transform this matrix into RREF:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right).$$

We see here that the left side of our augmented matrix cannot possibly equal I_3 (since there is a row of 0's). Hence the given matrix is not row equivalent to I_3, and is thus not invertible!

Section 2.3

11. Assume all of the following matrices are $n \times n$. Each part of this problem is an *implication* of the form "If "statement 1", then "statement 2"." Mark an implication as True if the truth of "statement 2" *always* follows whenever "statement 1" happens to be true. An implication is False if there is an instance in which "statement 2" is false but "statement 1" is true. Justify each answer.

    a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.
    b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.
    c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
    d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.
    e. If A^T is not invertible, then A is not invertible.

    Solutions.

    a. True. This follows from Theorem 8 since (b.) and (d.) are equivalent statements (they are either both true or both false).
    b. True. This follows from Theorem 8 since (e.) and (h.) are equivalent statements.
    c. False. This is claiming that the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times n$ matrix A – which we have seen not to be true (let the bottom row of A be a row of all zeros, then the equation definitely will not be consistent for each $\mathbf{b}$).
    d. True. This follows from the contrapositive statements of Theorem 8, i.e., since we know (c.) and (d.) are equivalent, then clearly we have: If (c.), then (d.). However, this means the contrapositive statement also holds: If not (d.), then not (c.).
    e. True. Once again this follows from the contrapositive statements of Theorem 8 since (a.) is equivalent to (l.).

33. T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1} if T is given by:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{pmatrix}.$$

Solution. We begin by determining the standard matrix A associated to the linear transformation T.

$$\begin{aligned} T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{pmatrix} \\ &= x_1 \begin{pmatrix} -5 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 9 \\ -7 \end{pmatrix}. \end{aligned}$$

Hence, $T(\mathbf{e}_1) = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ and $T(\mathbf{e}_2) = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$, which tells us that

$$A = \begin{pmatrix} -4 & 9 \\ 4 & -7 \end{pmatrix}.$$

Since A is a 2×2 matrix, we can use the determinant to determine if A, and hence T is invertible:

$$\det A = (-4)(-7) - (9)(4) = 28 - 36 = -8 \neq 0.$$

Since $\det A \neq 0$, we know that A is invertible, and hence so is T. Using the formula for the inverse of a 2×2 matrix we then find

$$A^{-1} = \frac{1}{-8} \begin{pmatrix} -7 & -4 \\ -9 & -4 \end{pmatrix} = \begin{pmatrix} \frac{7}{8} & \frac{1}{2} \\ \frac{9}{8} & \frac{1}{2} \end{pmatrix}.$$

Thus, we see

$$T^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{8}x_1 + \frac{1}{2}x_2 \\ \frac{9}{8}x_1 + \frac{1}{2}x_2 \end{pmatrix}.$$