Homework 2 Solutions

Sections 1.4-1.5 & 1.7-1.9

Section 1.4

18. Let B be given by

$$B = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{pmatrix}.$$

Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

Solution. We begin by writing B in its row echelon form:

$$\text{REF}(B) = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

(Note, we have chosen to forgo showing the row reductions for brevity.) We see from the REF above that B contains a pivot position in every row. Hence, by Theorem 4, the columns of B must span $\mathbb{R}^4$. Additionally, by the same theorem, since the columns of B span $\mathbb{R}^4$, the equation $B\mathbf{x} = \mathbf{y}$ has a solution for each $\mathbf{y}$ in $\mathbb{R}^4$.

Section 1.5

26. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution. We know the solution set to a nonhomogeneous equation will always look like

$$\mathbf{x} = \mathbf{p} + \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$$

for some vectors $\mathbf{p}$ and $\mathbf{v}_i$. We also know that the solution to the homogeneous equation will simply be $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Now, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution, then our solution set must simply be $\mathbf{x} = \mathbf{p}$, i.e., the "span-part" of the solution must be zero, otherwise we would have infinitely many solutions! Since the "span-part" is $\mathbf{0}$, then it must be the case that the solution to the homogeneous equation only has the solution $\mathbf{0}$ as well (since the "span-parts" of the homogeneous and nonhomogeneous equations must be the same). Hence, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Reversing this argument then shows that if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ must have a unique solution!

Section 1.7

12. Find the value(s) of h for which the vectors are linearly *dependent.* Justify your answer.

$$\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -6 \\ 7 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 8 \\ h \\ 4 \end{pmatrix}$$

Solution. In order to be linearly dependent we must be able to find weights x_1, x_2, x_3 (not all zero) such that

$$x_1 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -6 \\ 7 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ h \\ 4 \end{pmatrix} = \mathbf{0}.$$

In order to solve this vector equation, we write the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{array}\right),$$

and perform Gaussian elimination to put it in REF:

$$\left(\begin{array}{ccc|c} 2 & -6 & 8 & 0 \\ 0 & -5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From the REF we see the system is consistent and will have a free variable (x_3) for every value of h. Thus, for any h-value we can pick a nonzero value for x_3 and be assured that we have a nonzero solution to the above vector equation.

Hence, the given vectors are linearly dependent for every value of h!

39. Suppose A is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

Solution. Since $\mathbf{0}$ is an element of $\mathbb{R}^m$, then by the given assumptions $A\mathbf{x} = \mathbf{0}$ has at most one solution. We know that $A\mathbf{x} = \mathbf{0}$ always has the trivial solution, so it follows that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. The matrix equation $A\mathbf{x} = \mathbf{0}$ corresponds to the vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$, so this vector equation has only the trivial solution. But, this is exactly the definition of linear independence, so the columns of A must be linearly independent.

Section 1.8

10. Find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ where A is given by

$$A = \begin{pmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{pmatrix}.$$

Solution. Finding all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped to the zero vector by $\mathbf{x} \to A\mathbf{x}$ is equivalent to finding the solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$. Hence, to solve this we will row reduce (to RREF) the augmented matrix $(A \mid \mathbf{0})$ to determine our solutions:

$$\left(\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{array}\right) \sim \cdots \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, we see our solution, in parametric form, is

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$$

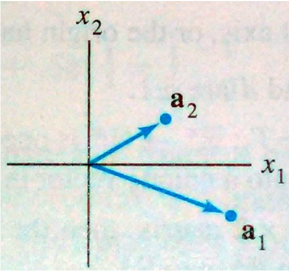
We also could have written our solution in parametric vector form as:

$$x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Section 1.9

28. Determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify your answer.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = [\mathbf{a}_1\ \mathbf{a}_2]$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are shown in the figure below:



Solution. (a) Determining whether T is one-to-one is equivalent to asking whether the columns of its standard matrix A are linearly independent. This means we want to know if $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a linearly independent set. We know that a set of two vectors is linearly dependent if and only if one vector is a scalar multiple of the other, and clearly from the given figure $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples of each other. Thus the columns of A are linearly independent and hence T is one-to-one.

(b) The easiest way to determine where T is onto here is by inspection. Ask yourself if there is any point in the plane (in $\mathbb{R}^2$) that you could not reach by using some linear combination of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ – clearly the answer here is no. (You could also think of creating a grid using the two vectors and filling in the squares between the gridlines – everything that is shaded will be in the span of your two vectors!) Thus, any $\mathbf{b}$ in $\mathbb{R}^2$ can be written as a linear combination of the columns of A, and hence T is onto.

35. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ can you give a relation between m and n? If T is one-to-one what can you say about m and n?

Solution. If T is onto then the standard matrix must be have a pivot in each row (since the augmented matrix with any given $\mathbf{b} \in \mathbb{R}^m$ must always be consistent). Having a pivot in every row means we must have $m \leq n$ (if we had more rows than columns there would be no way to have a pivot in each row – we would run out of pivot columns!).

If T is one-to-one then the columns of the standard matrix must be linearly independent. We know that a set containing more vectors than the dimension of the space (the codomain in this case) will always be linearly dependent. Hence, since the dimension of the codomain ($\mathbb{R}^m$) is m, and we have n columns in the standard matrix we must have $m \geq n$.