Homework 1 Solutions

Sections 1.1, 1.2 & 1.3

Section 1.1

14. Solve the system

$$\begin{cases} x_1 - 3x_2 = 5 \\ -x_1 + x_2 + 5x_3 = 2 \\ x_2 + x_3 = 0 \end{cases}$$

Solution. The linear system can be rewritten as an augmented matrix which we then row reduce as follows:

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Hence we see that the solution is $x_1 = 2, x_2 = -1$ and $x_3 = 1$, or $(2, -1, 1)$.

25. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\left(\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array}\right)$$

Solution. We begin by putting the matrix into row echelon form:

$$\left(\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+k+h \end{array}\right).$$

Thus, we see that, in order to be consistent, we must have $2g + k + h = 0$ (otherwise the bottom row will imply $0 = b$ where b is nonzero, which is impossible!).

Section 1.2

12. Find the general solution of the system whose augmented matrix is given below:

$$\left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right)$$

Solution. First, we write the augmented matrix in reduced row echelon form:

$$\left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We see that columns 1 and 3 are pivot columns, making x_1, x_3 basic variables, and x_2, x_4 free variables. We can thus represent the solution in parametric form

$$\begin{cases} x_1 = 7x_2 - 6x_4 + 5 \\ x_2 \text{ is free} \\ x_3 = 2x_4 - 3 \\ x_4 \text{ is free.} \end{cases}$$

18. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system:

$$\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 5 & h & -7 \end{array}\right)$$

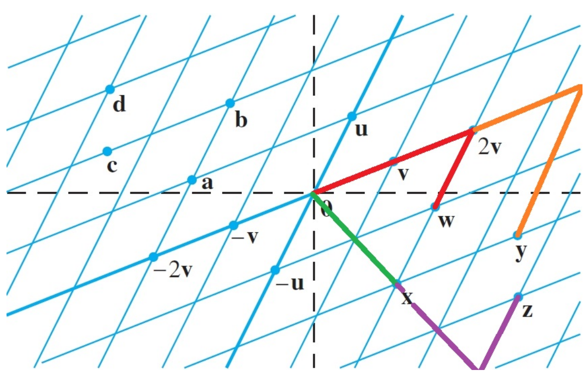
Solution. Putting the augmented matrix into row echelon form we see

$$\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 5 & h & -7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{array}\right).$$

Since there is a nonzero value in the bottom right corner we must have $h + 15 \neq 0$ for this system to be consistent (otherwise we would have $0 = b$ for some nonzero b). Therefore, we must have $h \neq -15$ for the system to be consistent.

Section 1.3

8. Use the figure below to write the vectors **w, x, y, z** as a linear combination of **u** and **v**. Is every vector in $\mathbb{R}^2$ a linear combination of **u** and **v**?



We can follow the colored lines in the above diagram to determine the linear combinations:

- $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$
- $\mathbf{x} = 2(\mathbf{v} - \mathbf{u}) = 2\mathbf{v} - 2\mathbf{u}$
- $\mathbf{y} = 3.5\mathbf{v} - 2\mathbf{u}$
- $\mathbf{z} = 4(\mathbf{v} - \mathbf{u}) + u = 4\mathbf{v} - 3\mathbf{u}$

10. Write a vector equation that is equivalent to the given system of equations:

$$\begin{cases} 4x_1 + x_2 + 3x_3 = 9 \\ x_1 - 7x_2 - 2x_3 = 2 \\ 8x_1 + 6x_2 - 5x_3 = 15 \end{cases}$$

Solution.

$$x_1 \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -7 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 15 \end{pmatrix}.$$

26. Let $A = \begin{pmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{pmatrix}$, let $\mathbf{b} = \begin{pmatrix} 10 \\ 3 \\ 3 \end{pmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is $\mathbf{b}$ in W?

b. Show that the third column of A is in W.

Solution.

a. We know that $\mathbf{b}$ is in W if and only if the corresponding augmented matrix is consistent. Putting the augmented matrix into row echelon form we see:

$$\left(\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We see from above that the augmented matrix corresponds to a consistent system, and thus $\mathbf{b}$ is in W.

b. Notice that

$$\begin{pmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix},$$

hence the third column of A is in W.

Note, we could have proceeded similarly to part (a.) to show that the third column was in W as well.