Math 129 Midterm 3 Study Guide

As is usually the case, this is not intended to be comprehensive but merely to serve as a guide to help make your study time more efficient.

1 Brief Review

Section 8.5 Work will be covered on this exam! For detailed explanation of how to do problems like this, see the solutions for Written Homework 4. Here are basic ideas to keep in mind.

- We often want to split the region into small section where the displacement (the amount we need to move that slice) is approximately constant.
- Then we add up the work on each of these slices.
- Since "Work=Force X Displacement", we need to find the force required to move the slice (usually the force required to counteract gravity). And then them multiply that by the distance the slice has to move.
- If specific variables aren't given, you can draw them in your figure wherever you want.
- To find Force, remember that weight (in lbs) is a force and that mass (in KG) time acceleration due to gravity ($9.8m/s^2$) gives you force in Newtons. (We only need to multiply by the gravitation constant if we are working in SI units.)

Chapter 9

This chapter covers sequences and series. Following the problems, there are two sheets of information describing the convergence tests used for improper integrals and series. Be sure to read over them to review the specific conditions of each of the convergence test.
Some specifics:

Section 9.1 A **sequence** is an infinite list of numbers. We often want to know if an sequence **converges**. This is the same as asking if the terms in the sequence get, and stay, as close as we want to some value. To find what a sequence $\{a_n\}$ converges to we find

$$\lim_{n\to\infty} a_n$$

Section 9.2 A **series** is what you get if you add up all the terms in a sequence. Example:

$$1, 1/2, 1/4, 1/8\dots$$

is a sequence since it is a list of terms.

$$1 + 1/2 + 1/4 + 1/8 + \cdots$$

is a series since it is a sum of an infinite lists of terms. A sequence or series is **geometric** if the ratio of any two consecutive terms is constant (same constant for any two consecutive terms).

A finite geometric series always converges. It converges to

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a - ax^n}{1 - x} = \frac{\text{first term - first omitted term}}{1 - \text{common ratio}} \qquad x \neq 1$$

An infinite geometric series converges if the absolute value of the common ratio is less than 1. In this case, the infinite series converges to

$$a + ax + ax^2 + ax^3 + \cdots = \frac{a}{1 - x} = \frac{\text{first term}}{1 - \text{common ratio}} \qquad |x| < 1.$$

If the absolute value of the ratio is greater than or equal to one, the infinite geometric sum diverges.

Section 9.3 In this section, we talk about convergence of series in general. We say that a series **converges** if its sequence of partial sums converges. Be sure to review the properties on page 507.

In particular, it is important to remember that if $\lim_{n\to\infty} a_n \neq 0$ or the limit of the terms does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. But!!!!, if $\lim_{n\to\infty} a_n = 0$, maybe the sum converges and maybe it doesn't. We need to do more tests to know which it is. (This is sometimes called the divergence test).

The integral test is introduced in this section. See that last two pages on this packet for more details. Be sure to remember the function $f(x)$ with $a_n = f(n)$ must be decreasing and positive in order to use the integral test. We also used the integral test to develop a p-series test for series.

Section 9.4 This section added to the giant list of techniques used to analyze series. These include the Comparison Test, the Limit Comparison Test, Absolute Convergence, the Ratio Test, and the Alternating Series Test. Specifics of these tests are on the last two pages of this packet. You should know the statements of each of these tests (hypotheses and conclusions) and several examples of typical series for which you would use them.

Section 9.5 This section introduces power series. You should be able to find an interval of convergence and a radius of convergence for a power series.

2 Problems

The following problems are in no particular order. They are designed to be difficult.

1. A spherical tank with radius 6 ft is filled up to 4 feet with gasoline. The top of the tank is 2 ft underground, and you would like to pump all of the gasoline to the surface. How much work will be required? (Remember that gasoline weighs 42 pounds per cubic foot.)

2. An 8-foot ladder is leaning against the wall with the top of the ladder 6 feet up. You decide to knock the ladder over before it starts to slide, because you hate sliding-ladder problems. Calculate the work done by gravity on the ladder, assuming that the ladder has a uniform density of 3 lb/ft.

3. You have filled your swimming pool with orange juice, and your landlord is furious! You need to pump all the juice out of the pool, and fast. You have a rectangular pool, 15 feet long, 20 feet wide, and 12 feet deep. It is filled to a depth of 10 feet. You never buy pulp-free orange juice, so the density of the liquid changes linearly from 70 lb/ft^3 at the bottom to 65 lb/ft^3 at the surface. Calculate the work necessary to pump all of the orange juice to the top of the pool.

4. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

5. The Riemann zeta-function ζ defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

   is used (among other things) to study the distribution of prime numbers. What is the domain of ζ? Explain.

6. A ball is dropped from a height of 6 feet and bounces. Each bounce is 2/3 the height of the previous.
   (a) Sketch a picture of the first 3 bounces. Find the total vertical distance traveled by the ball when it hits the ground the third time.
   (b) Find the total vertical distance traveled by the ball when it hits the ground the nth time.

7. Give an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges, while $\sum_{n=1}^{\infty} a_{n+1} - a_n$ converges. To receive credit, you must show that your example works.

8. Use the ratio test and the limit comparison test to determine whether the series

$$\sum_{n=0}^{\infty} \frac{n^2 + 3n - 2}{n^5 - 4^n + 11}$$

   converges. Which method do you prefer?

9. Find the interval of convergence, testing the endpoints of

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{\sqrt{n}}.$$

10. For each of the following power series, find the interval of convergence, and be sure to check the endpoints:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

11. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n$

(b) $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$

(c) $\displaystyle\sum_{n=0}^{\infty} c_n (-3)^n$

(d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

12. Determine whether the following series converge or diverge. State which convergence test you used.

A. $\sum_{n=1}^{\infty} 2^{-n}$

B. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n\sqrt{n-1}}}$

C. $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$

D. $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$

E. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^2}$

F. $\sum_{n=0}^{\infty} 2^n$

G. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

13. Suppose you know that $0 \le b_n \le \frac{1}{n} \le a_n$ and $0 \le c_n \le \frac{1}{n^2} \le d_n$ for all n.

A. Which of the series $\sum a_n, \sum b_n, \sum c_n,$ and $\sum d_n$ definitely converge? Justify your answer.

B. Which of the series $\sum a_n, \sum b_n, \sum c_n,$ and $\sum d_n$ definitely diverge? Justify your answer.

INTEGRALS AND SERIES

[7.7] Definition of convergence of improper integrals:

Suppose $f(x)$ is positive for $x \geq a$.

If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ is a finite number, we say that $\int_a^\infty f(x)\,dx$ **converges** and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Otherwise, we say that the integral **diverges**.

[7.8] Comparison Test for $\int_a^\infty f(x)\,dx$

Assume $f(x)$ is positive. Proving convergence or divergence involves two stages:

(1) By looking at the behavior of the integrand for large x, guess whether the integral converges or not.

(2) Confirm the guess by finding an appropriate function and inequality so that:

If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

[7.8] Useful Integrals for Comparison

(1) $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $1/(p-1)$ for $p > 1$ and diverges for $p \leq 1$.

(2) $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \geq 1$.

(3) $\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

[9.2] Infinite Geometric Series

If $|x| < 1$, $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$

[9.3] Connection between Series and Integrals – The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive for $x \geq c$.

If $\int_c^\infty f(x)\,dx$ converges, then $\sum a_n$ converges.

If $\int_c^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

[9.3] A Useful Series for Comparison

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

[9.4] Comparison Test

Suppose $0 \le a_n \le b_n$ for all n.

If $\sum b_n$ converges, then $\sum a_n$ converges.

If $\sum a_n$ diverges, then $\sum b_n$ diverges.

[9.4] Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

[9.4] Convergence of Absolute Value

If $\sum |a_n|$ converges, then so does $\sum a_n$.

[9.4] The Ratio Test

For a series $\sum a_n$, suppose the sequence of ratios $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$,

If $L < 1$, then $\sum a_n$ converges.

If $L > 1$ or if L is infinite, then $\sum a_n$ diverges.

If $L = 1$, the test does not tell us anything about the convergence of $\sum a_n$.

[9.4] Alternating Series Test

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges if $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$.

[9.5] Power Series – Radius of Convergence (ROC or R) and Interval of Convergence (IOC)

For the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$:

- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|$ is infinite, then $R = 0$ and the series converges only for $x = a$.
- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = 0$, then $R = \infty$ and the series converges for all values of x.
- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = K$, where K is finite and nonzero, then $R = 1/K$ and the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Courtesy of Faith Bridges