

Homework 6

Sections 10.1–10.3

1. Compute the degree 3 Taylor polynomial for $f(x) = \arcsin(x)$ about $x = 0$.

Solution. We begin by computing the first three derivatives of $f(x)$:

$$\begin{aligned}f(x) &= \arcsin x \\f'(x) &= \frac{1}{\sqrt{1-x^2}} \\f''(x) &= \frac{x}{(1-x^2)^{3/2}} \\f'''(x) &= \frac{3x^2}{(1-x^2)^{5/2}} + \frac{1}{(1-x^2)^{3/2}}.\end{aligned}$$

Hence, the degree three Taylor polynomial is given by

$$\begin{aligned}P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\&= 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 \\&= x + \frac{x^3}{6}.\end{aligned}$$

2. Compute the N th degree Taylor polynomial for $f(x) = \frac{3}{x-1}$ about $x = 0$. (Be smart here – try to use something we learned in Chapter 9.)

Solution. First we notice that our given function can be rewritten as:

$$f(x) = \frac{3}{x-1} = \frac{3}{-(1-x)} = \frac{-3}{1-x}.$$

Hence we see that $f(x)$ is the sum of an infinite geometric series with

$$a = -3 \quad \text{and} \quad 'x' = x.$$

Therefore,

$$f(x) = \frac{-3}{1-x} = \sum_{n=0}^{\infty} (-3)x^n,$$

and so the N th degree Taylor polynomial will be:

$$P_N(x) = \sum_{n=0}^N (-3)x^n.$$

3. Find the Maclaurin series for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ (Be sure to simplify!). Hint: Try using the properties of logarithms first! And be smart again – try using ideas from Section 10.3!

Solution. We begin by rewriting our function $f(x)$ as:

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

Now, we know (or could easily find) that the Taylor series for $\ln(x)$ about $x = 1$ is given by

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots.$$

Since we are finding the series for $f(x)$ about $x = 0$ (because we want the Maclaurin series), when x is near 0, $1-x$ and $1+x$ are both near 1 – hence we can make a substitution into the expansion for $\ln(x)$. Thus,

$$\begin{aligned} \ln(1+x) &= ((1+x)-1) - \frac{1}{2}((1+x)-1)^2 + \frac{1}{3}((1+x)-1)^3 - \frac{1}{4}((1+x)-1)^4 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \end{aligned}$$

and

$$\begin{aligned} \ln(1-x) &= ((1-x)-1) - \frac{1}{2}((1-x)-1)^2 + \frac{1}{3}((1-x)-1)^3 - \frac{1}{4}((1-x)-1)^4 + \dots \\ &= (-x) - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots. \end{aligned}$$

Subtracting these series we find

$$\begin{aligned} f(x) = \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - \left((-x) - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right) \\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots. \end{aligned}$$

4. Let $f(x)$ be a function whose degree 3 Taylor polynomial about $x = 2$ can be simplified to

$$P_3(x) = 5 + x - 3x^2 - 5x^3.$$

Is f increasing or decreasing at $x = 2$? Why?

Solution. Since Taylor polynomials provide “good” approximations to functions near a specific point, $P_3(x)$ will be a good approximation to $f(x)$ near $x = 2$, i.e., both $P_3(x)$ and $f(x)$ will behave similarly near $x = 2$. Thus, if $P_3(x)$ is increasing or decreasing at $x = 2$, then $f(x)$ will do the same.

We compute the derivative of $P_3(x)$ to be

$$P_3'(x) = 1 - 6x - 15x^2,$$

and note that $P_3'(2) = 1 - 6(2) - 15(2)^2 = -71$. Since $P_3'(2)$ is negative, we know $P_3(x)$ is decreasing at $x = 2$ and hence $f(x)$ is also decreasing at $x = 2$.