Homework 4 Solutions

Section 7.7

Determine whether the following improper integrals converge or diverge. You **must justify your conclusion** to receive credit!

1. $\displaystyle\int_1^\infty \frac{1}{x(1+e^x)}\,dx$

   Solution. **Converges!** We have

   $$\boldsymbol{\frac{1}{x(1+e^x)}} \leq \frac{1}{1+e^x} \leq \frac{1}{e^x} = e^{-x},$$

   and we know

   $$\int_0^\infty e^{-x}dx$$

   converges (since $a = 1 > 0$). Therefore, $\int_0^\infty \frac{1}{x(1+e^x)}\,dx$ converges.

2. $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

   Solution. **Converges!** We have

   $$\boldsymbol{\frac{\sqrt{x+1}}{x^2}} \leq \frac{\sqrt{x+x}}{x^2} = \frac{\sqrt{2x}}{x^2} = \frac{\sqrt{2}}{x^{3/2}},$$

   and we know

   $$\int_1^\infty \frac{\sqrt{2}}{x^{3/2}}\,dx = \sqrt{2}\int_1^\infty \frac{1}{x^{3/2}}\,dx$$

   converges (since $p = \frac{3}{2} > 1$). Thus, $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges.

3. $\displaystyle\int_0^\pi \frac{\sin x}{\sqrt{\pi - x}}\,dx$

   Solution. **Converges!** We begin by making the substitution $u = \pi - x$ so that $du = -dx$ and our integral becomes

   $$\int_0^\pi \frac{\sin x}{\sqrt{\pi - x}}\,dx = \int_\pi^0 \frac{\sin(\pi - u)}{\sqrt{u}}(-du) = \int_0^\pi \frac{\sin(\pi - u)}{\sqrt{u}}\,du.$$

   Now we want to use the comparison test for this "new" integral, so we see

   $$\boldsymbol{\frac{\sin(\pi - u)}{\sqrt{u}}} \leq \frac{1}{\sqrt{u}}.$$

   We know that

   $$\int_0^1 \frac{1}{\sqrt{u}}\,du = \int_0^1 \frac{1}{u^{1/2}}\,du$$

   converges (since $p = \frac{1}{2} < 1$). Thus, $\int_0^\pi \frac{1}{\sqrt{u}}\,du$ converges, and hence $\int_0^\pi \frac{\sin x}{\sqrt{\pi - x}}\,dx$ converges.

4. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,dx$

Solution. **Converges!** We have

$$\boldsymbol{\frac{1}{\sqrt{x^6+1}}} \leq \frac{1}{\sqrt{x^6}} = \frac{1}{x^3},$$

and we know (be sure to note the bounds of integration here)

$$\int_1^\infty \frac{1}{x^3}\,dx$$

converges (since $p = 3 > 1$). Thus, $\int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx$ converges and therefore it follows (since $\frac{1}{\sqrt{x^6+1}}$ is continuous on $[0,1]$) that $\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,dx$ converges.

5. $\displaystyle\int_3^\infty \frac{1}{2+\cos(x)+\ln(x)}\,dx$

Solution. **Diverges!** We have

$$\frac{1}{2x} = \frac{1}{x+x} \leq \frac{1}{3+x} \leq \frac{1}{3+\ln(x)} = \frac{1}{2+1+\ln(x)} \leq \boldsymbol{\frac{1}{2+\cos(x)+\ln(x)}},$$

and we know

$$\int_3^\infty \frac{1}{2x}\,dx = \frac{1}{2}\int_3^\infty \frac{1}{x}\,dx$$

diverges (since $p = 1$). Thus, $\int_3^\infty \frac{1}{2+\cos(x)+\ln(x)}$ diverges.

6. $\displaystyle\int_0^5 \frac{1}{\sqrt{x^5+x^3}}\,dx$

Solution. **Diverges!** We begin by considering the integral on a smaller interval:

$$\int_0^1 \frac{1}{\sqrt{x^5+x^3}}\,dx.$$

On this interval we have

$$\frac{1}{\sqrt{2x^3}} = \frac{1}{\sqrt{x^3+x^3}} \leq \boldsymbol{\frac{1}{\sqrt{x^5+x^3}}},$$

and we know

$$\int_0^1 \frac{1}{\sqrt{2x^3}}\,dx = \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{x^3}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{x^{3/2}}$$

diverges (since $p = \frac{3}{2} > 1$). Thus, $\int_0^1 \frac{1}{\sqrt{x^5+x^3}}\,dx$ diverges and hence $\int_0^5 \frac{1}{\sqrt{x^5+x^3}}\,dx$ diverges.

7. $\displaystyle\int_0^5 \frac{x}{\sqrt{x^5+x^3}}\,dx$

Solution. **Converges!** We begin by considering the integral on a smaller interval:

$$\int_0^1 \frac{x}{\sqrt{x^5+x^3}}\,dx.$$

On this interval we have

$$\frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x^5 + x^3}}} \leq \frac{x}{\sqrt{x^3}} = \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}},$$

and we know

$$\int_0^1 \frac{1}{x^{1/2}}\ dx$$

converges (since $p = \frac{1}{2} < 1$). Therefore, $\int_0^1 \frac{x}{\sqrt{x^5+x^3}}\ dx$ converges, and since the integrand is continuous on $[1,5]$ it follows that $\int_0^5 \frac{x}{\sqrt{x^5+x^3}}\ dx$ converges.