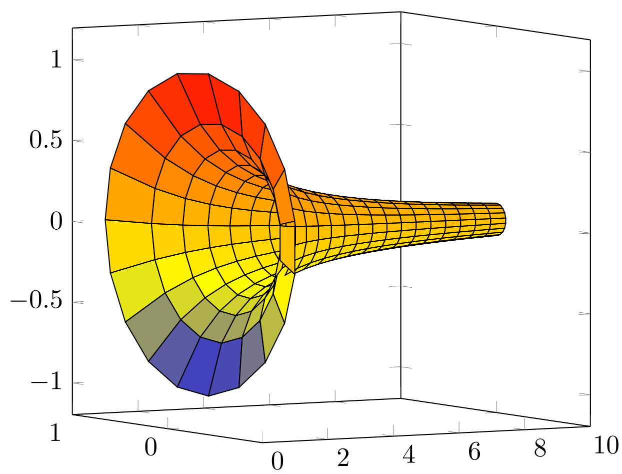
Homework 3 Solution

Sections 7.6

1. *Gabriel's horn* is the name given to the solid shown below, which is formed by revolving about the x-axis the unbounded region under the curve $y = \frac{1}{x}$ for $x \geq 1$ (we will see much more of this idea in Section 8.2).

Gabriel's horn on $1 \leq x \leq 10$



An interesting fact is that Gabriel's horn has a *finite* volume, but an *infinite* surface area. This would mean, for instance, that you can fill Gabriel's horn with a finite amount of paint, but it would take an infinite amount of paint to color its inside surface!

Use the following formulas for the volume and surface area of Gabriel's horn to show that the former is finite while the latter is infinite. (You may only use the techniques learned in Section 7.6, i.e., do not use comparisons from Section 7.7!)

$$\text{Volume } = \pi \int_1^\infty x^{-2} dx \qquad\qquad \text{Surface Area } = 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} dx$$

(Hints: The substitution $u = x^2$ might be helpful, and remember, $\frac{1}{x} = \frac{x}{x^2}$. You may also use that $\int \frac{\sqrt{x^2+1}}{x^2} dx = \frac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$.)

Solution. We being by showing that the volume of Gabriel's horn is finite. We are told that the volume is given by the improper integral:

$$V = \pi \int_1^\infty x^{-2} dx,$$

which we know to rewrite as

$$V = \lim_{b \to \infty} \pi \int_1^b x^{-2} dx.$$

Next, we simply compute this definite integral and take the limit in the last step:

$$\begin{aligned} V &= \lim_{b\to\infty} \pi \int_1^b x^{-2} dx \\ &= \lim_{b\to\infty} \pi \left(-\frac{1}{x}\Big|_1^b \right) \\ &= \lim_{b\to\infty} \pi \left(-\frac{1}{b} + 1 \right) \\ &= \pi (0+1) \\ &= \pi. \end{aligned}$$

Hence, the volume of Gabriel's horn is finite!

Next, we would like to show the surface area of Gabriel's horn is infinite. Again, we are told the surface area is given by the improper integral:

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} dx,$$

which we rewrite as

$$SA = \lim_{b\to\infty} 2\pi \int_1^b \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} dx.$$

The hint now reminds us that $\frac{1}{x} = \frac{x}{x^2}$, so we make this substitution,

$$SA = \lim_{b\to\infty} 2\pi \int_1^b \frac{x}{x^2}\sqrt{\frac{x^4+1}{x^4}} dx.$$

Also, as per the hint, we make a substitution, $u = x^2$ so that $du = 2xdx$, and hence $\frac{1}{2}du = xdx$, to see

$$\begin{aligned} SA &= \lim_{b\to\infty} 2\pi \int_1^b \frac{x}{x^2}\sqrt{\frac{x^4+1}{x^4}} dx \\ &= \lim_{b\to\infty} \pi \int_1^{b^2} \frac{1}{u}\sqrt{\frac{u^2+1}{u^2}} ux \\ &= \lim_{b\to\infty} \pi \int_1^{b^2} \frac{\sqrt{u^2+1}}{u^2} du. \end{aligned}$$

Again, by the hint, we know the value of this integral:

$$\begin{aligned} \lim_{b\to\infty} \pi \int_1^{b^2} \frac{\sqrt{u^2+1}}{u^2} du &= \lim_{b\to\infty} \pi \left[\left(-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right) \right) \Big|_1^{b^2} \right] \\ &= \lim_{b\to\infty} \pi \left[\left(-\frac{\sqrt{b^4+1}}{b^2} + \ln\left(b^2 + \sqrt{b^4+1}\right) \right) - \left(-\sqrt{2} + \ln\left(1+\sqrt{2}\right) \right) \right] \\ &= \infty. \end{aligned}$$

Where the last equality follows since $\lim_{b\to\infty} \ln(b^2 + \sqrt{b^4+1}) = \infty$. Therefore, we have that the surface area of Gabriel's horn is infinite!