Homework 2 Solutions

Sections 7.3 & 7.4

1. Determine
$$\int \frac{\cos(x)}{\sin^2(x) - \sin(x) - 2}\, dx.$$

   Solution. We begin by making a substitution; let $u = \sin(x)$ then $du = \cos(x)dx$, so our integral can be rewritten as
$$\int \frac{\cos(x)}{\sin^2(x) - \sin(x) - 2}\, dx = \int \frac{1}{u^2 - u - 2}\, du.$$
   Factoring we see,
$$\int \frac{1}{u^2 - u - 2}\, du = \int \frac{1}{(u-2)(u+1)}\, du.$$
   Now, we will use partial fractions to rewrite the integrand:
$$\frac{1}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1} = \frac{A(u+1) + B(u-2)}{(u-2)(u+1)}.$$
   So (using our short-cut) we have that $1 = A(u+1) + B(u-2)$, and evaluating this at $u = -1$ gives $B = -\frac{1}{3}$, and evaluating at $u = 2$ we find $A = \frac{1}{3}$. (Note, we also could have distributed, combined like-terms, equated coefficients and solved a system of equations to find these values for A and B.)

   With these values of A and B we now have
$$\begin{aligned}\int \frac{1}{(u-2)(u+1)}\, du &= \int \frac{\frac{1}{3}}{u-2} + \frac{-\frac{1}{3}}{u+1}\, du \\ &= \frac{1}{3}\ln|u-2| - \frac{1}{3}\ln|u+1| + C \\ &= \frac{1}{3}\ln|\sin(x) - 2| - \frac{1}{3}\ln|\sin(x) + 1| + C.\end{aligned}$$

2. Use completing the square to determine the following integral
$$\int \sqrt{-2x^2 + 16x - 23}\, dx.$$

   Hint: Use u-substitution followed by a trigonometric substitution *after* you complete the square.

   Solution. First, we will perform completing the square on the quadratic inside the radical:
$$\begin{aligned}-2x^2 + 16x - 23 &= -2(x^2 - 8x) - 23 \\ &= -2\left(x^2 - 8x + \left(\frac{-8}{2}\right)^2 - \left(\frac{-8}{2}\right)^2\right) - 23 \\ &= -2(x^2 - 8x + (-4)^2) + 2(-4)^2 - 23 \\ &= -2(x-4)^2 + 32 - 23 \\ &= -2(x-4)^2 + 9.\end{aligned}$$

Thus, we have

$$\begin{aligned}
\int \sqrt{-2x^2+16x-23}\;dx &= \int \sqrt{-2(x-4)^2+9}\;dx \\
&= \int \sqrt{9-2(x-4)^2}\;dx \\
&= \int \sqrt{2\left(\left(\frac{9}{2}\right)-(x-4)^2\right)}\;dx \\
&= \sqrt{2}\int \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-(x-4)^2}\;dx.
\end{aligned}$$

This looks close very close to the form we need for trigonometric substitution, so (as the hint suggests) let's make a substitution. Let $u=x-4$, then $du=dx$ and our integral can be rewritten as

$$\sqrt{2}\int \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-(x-4)^2}\;dx = \sqrt{2}\int \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-u^2}\;du.$$

Now (again as the hint suggests) let's make a trigonometric substitution. Let $u=\frac{3}{\sqrt{2}}\sin(\theta)$ so $du=\frac{3}{\sqrt{2}}\cos(\theta)\;d\theta$. Thus,

$$\begin{aligned}
\sqrt{2}\int \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-u^2}\;du &= \sqrt{2}\int \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-\left(\frac{3}{\sqrt{2}}\sin(\theta)\right)^2}\;\frac{3}{\sqrt{2}}\cos(\theta)\;d\theta \\
&= 3\int \frac{3}{\sqrt{2}}\sqrt{1-\sin^2(\theta)}\cos(\theta)\;d\theta \\
&= \frac{9}{\sqrt{2}}\int \cos^2(\theta)\;d\theta \\
&= \frac{9}{\sqrt{2}}\left(\frac{1}{2}\cos(\theta)\sin(\theta)+\frac{1}{2}\theta+C\right) \\
&\qquad \text{(We've computed this integral in class – check your notes!)} \\
&= \frac{9}{2\sqrt{2}}(\cos(\theta)\sin(\theta)+\theta)+C.
\end{aligned}$$

Now, we know that $u=\frac{3}{\sqrt{2}}\sin(\theta)$ so $\frac{\sqrt{2}u}{3}=\sin(\theta)$ and $\theta=\arcsin\left(\frac{\sqrt{2}u}{3}\right)$. Using this and constructing a right triangle we can easily see $\cos(\theta)=\frac{\sqrt{9-2u^2}}{3}$. Hence,

$$\frac{9}{2\sqrt{2}}(\cos(\theta)\sin(\theta)+\theta)+C = \frac{9}{2\sqrt{2}}\left(\left(\frac{\sqrt{9-2u^2}}{3}\right)\left(\frac{\sqrt{2}u}{3}\right)+\arcsin\left(\frac{\sqrt{2}u}{3}\right)\right)+C.$$

Finally, we know that $u=x-4$, so we substitute this back in to find our final answer,

$$\frac{9}{2\sqrt{2}}\left(\left(\frac{\sqrt{9-2(x-4)^2}}{3}\right)\left(\frac{\sqrt{2}(x-4)}{3}\right)+\arcsin\left(\frac{\sqrt{2}(x-4)}{3}\right)\right)+C.$$