Homework 1 Solutions

Sections 7.1 & 7.2

1. Suppose that f is a continuous function, defined for all x, and that the values of the following integrals are known:

$$\int_0^1 f(x)\ dx = 5; \quad \int_{-1}^1 f(x)\ dx = 3; \quad \int_0^2 f(x)\ dx = 8; \quad \int_0^4 f(x)\ dx = 11.$$

Evaluate the following integrals:

(a) $\displaystyle\int_0^2 f(2x)\ dx$,

(b) $\displaystyle\int_0^\pi \sin(x)f(\cos(x))\ dx$,

(c) $\displaystyle\int_2^3 xf(8-x^2)\ dx$.

Solution.

(a) Let $u = 2x$, then $du = 2dx$ and hence $\frac{1}{2}du = dx$. Making this substitution,

$$\int_0^2 f(2x)\ dx = \int_0^4 f(u)\frac{1}{2}du = \frac{1}{2}\int_0^4 f(u)\ du = \frac{1}{2}(11) = \frac{11}{2}.$$

(b) Let $u = \cos(x)$, then $du = -\sin(x)dx$ and hence $-du = \sin(x)dx$. Making this substitution,

$$\int_0^\pi \sin(x)f(\cos(x))\ dx = \int_1^{-1} f(u)(-du) = -\int_1^{-1} f(u)\ du = \int_{-1}^1 f(u)\ du = 3.$$

(c) Let $u = 8 - x^2$, then $du = -2xdx$ and so $-\frac{1}{2}du = xdx$. Making this substitution,

$$\int_2^3 xf(8-x^2)\ dx = \int_4^{-1} f(u)\left(-\frac{1}{2}\right)du = -\frac{1}{2}\int_4^{-1} f(u)\ du = \frac{1}{2}\int_{-1}^4 f(u)\ du.$$

Now, we must break up this last integral into several pieces so that we can use the values of the given definite integrals:

$$\frac{1}{2}\int_{-1}^4 f(u)\ du = \frac{1}{2}\left(\int_{-1}^1 f(u)\ du - \int_0^1 f(u)\ du + \int_0^4 f(u)\ du\right) = \frac{1}{2}(3-5+11) = \frac{9}{2}.$$

2. Compute $\int \ln(x^2+1)\ dx$.

   Hint: $1 - \frac{1}{x^2+1} = \frac{x^2}{x^2+1}$.

   Solution.
   Let $u = \ln(x^2+1)$ and $dv = dx$, then $du = \dfrac{2x}{x^2+1}dx$ and $v = x$. So, using integration by parts, we have

$$\int \ln(x^2+1)\ dx = \big(\ln(x^2+1)\big)\,(x) - \int \frac{2x^2}{x^2+1}\ dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}dx.$$

   We now use the given hint to write the above as:

$$x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}dx = x\ln(x^2+1) - 2\int 1 - \frac{1}{x^2+1}dx.$$

   Finally, we can compute the integral,

$$\begin{aligned} x\ln(x^2+1) - 2\int 1 - \frac{1}{x^2+1}dx &= x\ln(x^2+1) - 2\,(x - \arctan(x) + C) \\ &= x\ln(x^2+1) - 2x + 2\arctan(x) + C. \end{aligned}$$

3. Compute $\int \sin^2(x)\ dx$.

   Hint: Use integration by parts first, then use the identity $\cos^2(x) = 1 - \sin^2(x)$.

   Solution.
   We begin by rewriting the integral,

$$\int \sin^2(x)\ dx = \int \sin(x)\sin(x)\ dx.$$

   Now, let $u = \sin(x)$ and $dv = \sin(x)dx$, then $du = \cos(x)dx$ and $v = -\cos(x)$. Using integration by parts,

$$\begin{aligned} \int \sin^2(x)\ dx = \int \sin(x)\sin(x)\ dx &= -\sin(x)\cos(x) - \int -\cos^2(x)dx \\ &= -\sin(x)\cos(x) + \int \cos^2(x)\ dx. \end{aligned}$$

   Since we know $\cos^2(x) = 1 - \sin^2(x)$, we rewrite the above as

$$\begin{aligned} \int \sin^2(x)\ dx &= -\sin(x)\cos(x) + \int 1 - \sin^2(x)\ dx \\ &= -\sin(x)\cos(x) + \int 1\ dx - \int \sin^2(x)\ dx. \end{aligned}$$

Notice that $\int \sin^2(x)\ dx$ appears on both sides of the equation above! Adding $\int \sin^2(x)\ dx$ to both sides yields

$$\begin{aligned} 2\int \sin^2(x)\ dx &= -\sin(x)\cos(x) + \int 1\ dx \\ &= -\sin(x)\cos(x) + x + C. \end{aligned}$$

Thus, dividing both sides by 2, we find

$$\int \sin^2(x)\ dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2} + C.$$