11.6 Applications and Modeling

Throughout this course many of the functions used in application and modeling problems have been given without any justification for their correctness. In some cases these functions are constructed from experimental data and in other cases a more rigorous theoretical approach is used. In this section we want to explore some examples where the theoretical approach is taken.

How a Layer of Ice Forms

When ice forms on a lake, the water on the surface freezes first. As heat from the water travels up through the ice and is lost to the air, more ice is formed.

Question: How thick is the layer of ice as a function of time?

A Differential Equation for the Thickness of the Ice

Suppose y represents the thickness of the ice as a function of time t. Since the thicker the ice the longer it takes the heat to get through, we will assume that the rate at which ice is formed is inversely proportional to the thickness. In other words, we assume that for some constant k,

Rate thickness is increasing =

which we can rewrite as

For this problem we will also have an initial condition,

Initial Condition:

This initial value problem can now be solved to find

$$y = \sqrt{2kt}.$$

The Net Worth of a Company

Consider a company whose revenues are proportional to its net worth (like interest on a bank account) but that must also make payroll payments. Under what circumstances does the company make money, and under what circumstances does it lose money?

We assume that revenue is earned continuously and that payments are made continuously. We also assume that the only factors affecting net worth are revenue and payroll.

Example A company's revenue is earned at a continuous rate of 5% of its net worth. At the same time, the company's payroll obligations amount to \$200 million a year, paid out continuously.

1. Write a differential equation that governs the net worth of the company, E in millions of dollars.

2. Solve the differential equation, assuming an initial net worth of W_0 million dollars.

3. What is the equilibrium solution? Is it stable or unstable?

The Velocity of a Falling Body: Terminal Velocity

When a sky-diver first jumps out of a plane his velocity is zero. The pull of gravity then makes his velocity increase. As the sky-diver speeds up, the air resistance also increases. Since the air resistance partly balances the pull of gravity, the force causing him to accelerate decreases. Thus, the velocity is an increasing function of time, but it is increasing at a decreasing rate (hence concave down). The air resistance increases until it balances gravity, when the sky-diver's velocity levels off.

A Differential Equation: Air Resistance Proportional to Velocity

In order to compute the velocity function we need to know exactly how air resistance depends on velocity. To determine whether air resistance is, say, proportional to the velocity, or is some other function of velocity, either lab experiments or a theoretical idea of how the air resistance is created is needed. For our purposes, let us assume that air resistance is proportional to velocity. Thus we have the following,

Net Force on Object:

$$\begin{aligned} F &= mg - kv \\ mg &= \text{Gravitational Force} \\ kv &= \text{Air Resistance} \end{aligned}$$

Newton's Second Law of Motion:

$$\begin{aligned} \text{Force} &= \text{Mass} \times \text{Acceleration} \\ F &= m \times \end{aligned}$$

Putting these together we have

This differential equation can be solved by separation of variables to find

$$v - \frac{mg}{k} = Ae^{-kt/m},$$

where A is an arbitrary constant.

Since the object starts from rest, $v(0) = 0$ and hence

$$\boxed{A = \qquad\qquad}$$

Therefore,

$$\boxed{v = \qquad\qquad\qquad\qquad}$$

Compartmental Analysis: A Reservoir

Many processes can be modeled as a container with various solutions flowing in and out (like the pollutants in a lake). We consider an example of a city's water reservoir, fed partly by clean water from a spring and partly by run-off from the surrounding land.

In New England and many other areas with much snow in the winter, the run off contains salt that has been put on the roads to make them safe for driving. We consider the concentration of salt in the reservoir.

A Differential Equation for Salt Concentration

Suppose we have the following:

- A water reservoir that holds 100 million gallons of water.
- The reservoir supplies a city with 1 million gallons a day.
- The reservoir is partly refilled by a spring that provides 0.9 million gallons a day, the remaining 0.1 million gallons comes from run-off.
  - The spring is clean water.
  - The run-off contains salt with a concentration of 0.0001 pounds per gallon.
- There was no salt in the reservoir initially, and the water is well mixed.

Find the concentration of salt in the reservoir as a function of time.

We will use the following notation:

Q	$=$	total quantity of salt (in pounds) in the reservoir
C	$=$	concentration of salt (in pounds/gallon)

We find that we can represent the concentration as

$$\boxed{C = \qquad\qquad\qquad\qquad\qquad\qquad}$$

We will find Q first, and then C. We know that

$$\boxed{\text{Rate of change of quantity of salt} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

So we want to determine these quantities. We see,

$$\boxed{\text{Rate salt entering} \quad = \qquad\qquad\qquad\qquad\qquad\qquad}$$

Similarly,

Rate salt leaving $=$

Therefore, Q satisfies the differential equation

Since there is no salt initially we have an initial condition of $Q(0) = 0$, so we solve this initial value problem to find

$$Q = 1000(1 - e^{-0.01t}).$$

Hence,

Concentration $= C =$