11.5 Growth and Decay

Consider the population of a region. If there is no immigration or emigration, the rate at which the population is changing is often proportional to the population. In other words, the larger the population, the faster it is growing because there are more people to have babies.

If the population at time t is P and its continuous growth rate is 2% per unit time, then we know

Rate of growth of population =

which we can rewrite as

The 2% growth rate is called the **relative growth rate** to distinguish it from the **absolute growth rate**, $\frac{dP}{dt}$. Notice that they measure different quantities. Since

Relative growth rate = 2% =

the relative growth rate is a percent per unit time, while

$$\text{Absolute growth rate} = \text{Rate of change of population} = \frac{dP}{dt}$$

is a change in population per unit time.

We can easily use separation of variables to find that the general solution to this differential equation is given by

$$P = P_0 e^{0.02t}.$$

Every solution to the equation

can be written in the form

where P_0 is the value of P at $t = 0$, and **$k > 0$ represents growth**, whereas **$k < 0$ represents decay**.

Recall that the **doubling time** of an exponentially growing quantity is the time required for it to double. Likewise, the **half-life** of an exponentially decaying quantity is the time required for half of it to decay.

Continuously Compound Interest

At a bank, continuous compounding means that interest is accrued at a rate that is a fixed percentage of the balance at that moment. Thus, the larger the balance, the faster interest is earned an the faster the balance grows.

Example A bank account earns interested continuously at a rate of 5% of the current balance per year. Assume that the initial deposit is \$1000 and that no other deposits or withdrawals are made.

1. Write the differential equation satisfied by the balance in the account.

2. Solve the differential equation and graph the solution.

The Difference Between Continuous and Annual Percentage Growth Rates

$P = P_0(1+r)^t$ $\Updownarrow$ r is the **annual growth rate**	$P = P_0e^{kt}$ $\Updownarrow$ k is the **continuous growth rate**

- The constant k in the differential equation $dP/dt = kP$ is the continuous growth rate.

Converting Continuous Rate to Annual Rate

In our first example, with a continuous interest rate of 5%, we obtain a balance of $B = B_0e^{0.05t}$, where t is time in years. At the end of one year the balance is $B_0e^{0.05}$. In that one year, our balance changed from B_0 to $B_0e^{0.05}$, that is, by a factor of $e^{0.05} = 1.0513$. Thus the annual growth rate is 5.13%. This is what the bank means when it says "5% compounded continuously for an effective annual yield of 5.13%". Since $P_0e^{0.05t} = P_0(1.0513)^t = P_0(1 + 0.0513)^t$ we have two different ways to represent the same function.

Converting Annual Rate to Continuous Rate

Since most growth is measured over discrete time intervals, a continuous growth rate is an idealized concept. A demographer who says a population is growing at the rate of 2% per year usually means that after t years the population is $P = P_0(1.02)^t$. To find the continuous growth rate, k, we express the population as $P = P_0e^{kt}$. At the end of one year, $P = P_0e^k$, so $e^k = 1.02$. Thus, $k = \ln 1.02 \approx 0.0198$; hence the continuous growth rate is $k = 1.98\%$, which is close to the annual growth rate of 2%, but definitely not the same. So, again we have two different representations of the same function since $P_0(1 + 0.02)^t = P_0(1.02)^t = P_0e^{0.0198t}$.

Pollution in the Great Lakes

In the 1960s pollution in the Great Lakes became an issue of public concern. We can set up a model for how long it would take for the lakes to flush themselves clean, assuming no further pollutants are being dumped into the lakes.

Suppose we have the following:

$$\begin{aligned} V &= \text{volume of lake} \\ Q &= \text{total quantity of pollutant in lake at time } t \\ r &= \text{rate at which clean water flows into the lake} \end{aligned}$$

We also suppose that water follows out of the lake at the rate r; that the pollutant is evenly spread throughout the lake; and that the clean water coming into the lake immediately mixes with the rest of the water.

Setting Up a Differential Equation for the Pollution

To model how Q changes with time, we write an equation for the rate at which Q changes. We know that

Rate Q Changes $= -$

where the negative sign represents the fact that Q is decreasing.

Now, at any time t,

$\frac{Q}{V} =$

Thus,

Rate pollutants leave in outflow $=$ $\qquad\qquad\qquad \times \qquad\qquad\qquad =$

Putting this all together we find the differential equation

which has the general solution

$$Q = Q_0 e^{-rt/V}.$$

The following table contains values of r and V for four of the Great Lakes.

	V (thousands of km^3)	r (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

Example According to the model, how long will it take for 90% of the pollution to be removed from Lake Erie?

Newton's Law of Heating and Cooling

Newton proposed that the temperature of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings. Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings.

Example When a murder is committed, the body, originally at 37°C, cools according to Newton's Law of Cooling. Suppose that after two hours the body is 35°C and the temperature of the surrounding air is 20°C.

1. Find the temperature, H, of the body as a function of t, the time in hours since the murder was committed.

2. If the body is found at 4pm at a temperature of 30°C, when was the murder committed?

More on Equilibrium Solutions

- An **equilibrium solution** is constant for all values of the independent variable. The graph is a horizontal line.
- An equilibrium is **stable** if a small change in the initial conditions gives a solution that tends toward the equilibrium as the independent variable tends to positive infinity.
- An equilibrium is **unstable** if a small change in the initial conditions gives a solution curve that veers away from the equilibrium as the independent as the independent variable tends to positive infinity.

- Solutions that do not veer away from an equilibrium solution are also called stable.
- If the differential equation is of the form $y' = f(y)$, equilibrium solutions can be found by setting $y' = 0$ and solving for y.

Example Find the equilibrium solution to the differential equation $dP/dt = -50+4P$ and determine whether the equilibrium is stable or unstable.