Homework 6 Solutions

Section 11.4

1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x+1}{2y}, \quad y(0) = 1.$$

Solution. Using separation of variables we rewrite our differential equation as

$$2ydy = (x+1)dx.$$

Now, we integrate both sides

$$\int 2ydy = \int (x+1)dx,$$

which gives

$$y^2 = x^2 + x + C.$$

Thus,

$$y = \pm\sqrt{x^2 + x + C}.$$

Using the initial condition that $y(0) = 1$ we find

$$1 = \pm\sqrt{0^2 + 0 + C},$$

which implies that the $\pm$ must be $+$ (since 1 is positive) and our $C = 1$. Hence, the solution to this initial value problem is

$$y = \sqrt{x^2 + x + 1}.$$

2. Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{\cos y}.$$

Solution. Using separation of variables we rewrite our differential equation as

$$\cos y \; dy = x\sqrt{x^2+1} \; dx.$$

Integrating both sides,

$$\int \cos y \; dy = \int x\sqrt{x^2+1} \; dx.$$

Note, for the integral on the right side we need to use a substitution of $u = x^2 + 1$, and upon integrating we find

$$\sin y = \frac{1}{3}(x^2+1)^{3/2} + C.$$

Solving this for y we find

$$y = \arcsin\left(\frac{1}{3}(x^2+1)^{3/2} + C\right).$$

3. Find the general solution to the differential equation

$$\frac{dy}{dx} + y + \frac{1}{y} = 0.$$

Solution. We begin by rewriting our equation as

$$\frac{dy}{dx} = -y - \frac{1}{y},$$

and we obtain a common denominator on the right side

$$\frac{dy}{dx} = -\frac{y^2+1}{y}.$$

Now, separating our variables,

$$\frac{y}{y^2+1} dy = -dx,$$

and integrating both sides,

$$\int \frac{y}{y^2+1}\, dy = -\int dx.$$

We need to use a substitution of $u = y^2 + 1$ on the left side, and upon integrating we find

$$\frac{1}{2} \ln \left| y^2 + 1 \right| = -x + C.$$

Simplifying,

$$\ln(y^2+1) = -2x + C,$$

and rewriting in exponential form,

$$y^2 + 1 = e^{-2x+C}.$$

Rewriting once again,

$$y^2 = Ce^{-2x} - 1,$$

and taking square-roots we find

$$y = \pm\sqrt{Ce^{-2x} - 1}.$$