Homework 5 Solutions

Sections 9.5 & 10.1

1. Determine the interval of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{n(n+1)}.$$

Solution. We proceed using the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|\frac{2^{n+1}(x-3)^{n+1}}{(n+1)(n+2)}\right|}{\left|\frac{2^n(x-3)^n}{n(n+1)}\right|} &= \lim_{n\to\infty} \frac{2^{n+1}n(n+1)|(x-3)^{n+1}|}{2^n(n+1)(n+2)|(x-3)^n|} \\
&= \lim_{n\to\infty} \frac{2n|(x-3)|}{(n+2)} \\
&= |x-3| \lim_{n\to\infty} \frac{2n}{n+2} \\
&= 2|x-3|.
\end{aligned}$$

Hence, $R = \dfrac{1}{2}$ and so our open interval of convergence is

$$\frac{5}{2} < x < \frac{7}{2}.$$

We now need to determine whether we have convergence at the endpoints of our interval. First, let $x = \frac{5}{2}$, then the series can be written

$$\sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)},$$

which is easily seen to converge using the alternating series test.
Letting $x = \frac{7}{2}$ our series becomes

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

which can be seen to converge via the comparison test (comparing with $\sum \frac{1}{n^2}$).
Thus, the interval of convergence for our series is

$$\frac{5}{2} \leq x \leq \frac{7}{2}.$$

2. Compute the degree three Taylor polynomial for $f(x) = \arcsin(x)$ about $x = 0$.

Solution. We begin by computing the first three derivatives of $f(x)$:

$$\begin{aligned}
f(x) &= \arcsin x \\
f'(x) &= \frac{1}{\sqrt{1-x^2}} \\
f''(x) &= \frac{x}{(1-x^2)^{3/2}} \\
f'''(x) &= \frac{3x^2}{(1-x^2)^{5/2}} + \frac{1}{(1-x^2)^{3/2}}.
\end{aligned}$$

Hence, the degree three Taylor polynomial is given by

$$\begin{aligned}
P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
&= x + \frac{x^3}{6}.
\end{aligned}$$

3. Compute the degree five Taylor polynomial for $f(x) = e^{-x}$ about $x = 1$.

Solution. We begin by computing the first five derivatives of $f(x)$:

$$\begin{aligned}
f(x) &= e^{-x} \\
f'(x) &= -e^{-x} \\
f''(x) &= e^{-x} \\
f'''(x) &= -e^{-x} \\
f^{(4)}(x) &= e^{-x} \\
f^{(5)}(x) &= -e^{-x}.
\end{aligned}$$

Hence, the degree five Taylor polynomial about $x = 1$ is given by:

$$\begin{aligned}
P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
&= e^{-1} - e^{-1}(x-1) + \frac{e^{-1}}{2!}(x-1)^2 - \frac{e^{-1}}{3!}(x-1)^3 + \frac{e^{-1}}{4!}(x-1)^4 - \frac{e^{-1}}{5!}(x-1)^5 \\
&= \frac{1}{e} - \frac{x-1}{e} + \frac{(x-1)^2}{e \cdot 2!} - \frac{(x-1)^3}{e \cdot 3!} + \frac{(x-1)^4}{e \cdot 4!} - \frac{(x-1)^5}{e \cdot 5!}.
\end{aligned}$$