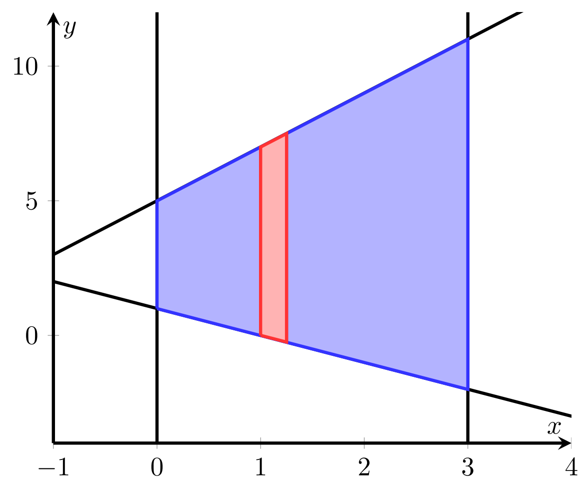
Homework 3 Solutions

Sections 8.2 & 8.5

1. The base of a solid is the region in the xy-plane bounded by the y-axis and the lines $y = 1 - x$, $y = 2x + 5$ and $x = 3$. Each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid.

   Solution.
   Graphing the given equations we see the base of our solid will be given by:

   We slice this region into vertical strips of width Δx and compute the volume of the object having this strip as a base.

   To compute the volume of this slice we first need to compute the area of the vertical equilateral triangle sticking up from this slice, then simply multiply by the width, Δx.

   The length of the base (and hence of all of the sides) at x of the equilateral triangle will be given by $(2x + 5) - (1 - x) = 3x + 4$. Now, recall that the area of an equilateral triangle is given by

   $$\text{Area} = \frac{\sqrt{3}}{4}(\text{Length of Side})^2.$$

   (Try deriving this formula – it's not too hard!) So the area of our triangle is:

   $$\text{Area} = \frac{\sqrt{3}}{4}(3x + 4)^2,$$

   and hence the volume of our slice is given by

   $$\text{Volume of Slice} = \frac{\sqrt{3}}{4}(3x + 4)^2 \Delta x.$$

   Thus, to compute the total volume of the object we just need to integrate:

   $$\int_0^3 \frac{\sqrt{3}}{4}(3x + 4)^2 \, dx = \frac{237\sqrt{3}}{4} \approx 102.6.$$

2. A 30 lb ball hangs at the bottom of a cable that is 50 ft long and weighs 20 lb (assume the cable has a constant density). The entire length of cable hangs over a cliff. Find the work done to raise the cable and get the ball to the top of the cliff.

Solution. We break this problem into two parts: the work required to raise the cable, and the work required to raise the ball.

To raise the cable we first should compute the density of it. Since we may assume the cable has a constant density and it is 50 ft long weighing 20 lb the density of the cable is $\frac{20}{50} = \frac{2}{5}$ lb/ft.

Now, let's consider an arbitrary portion of the cable that is Δy ft long and y ft from the top of the cliff. Using the density we computed the weight of this portion of the cable is $\frac{2}{5}\Delta y$ lb. (Note, since we are talking about weight, we do not need to multiply by the acceleration due to gravity since weight = gravitational force.) Thus we have a force of $\frac{2}{5}\Delta y$ and we need to move it y feet, which means the work done on this small portion is:

$$\text{Work on a Portion of Cable} \approx \frac{2}{5} y \Delta y \text{ ft-lb.}$$

Thus, the work done in raising *only the cable* will be given by

$$\text{Work Done on Cable} = \int_0^{50} \frac{2}{5} y \; dy \text{ ft-lb.}$$

Next, we need to compute the work done in raising the ball. Since the weight of the ball is 30 lb, the gravitational force on the ball is 30 lb. We must raise the ball 50 ft and so the work done in raising *only the ball* will be given by

$$\text{Work Done on Ball} = 30 \cdot 50 = 1500 \text{ ft-lb.}$$

Adding these values we find

$$\begin{aligned} \text{Total Work Done} &= \text{Work Done on Cable} + \text{Work Done on Ball} \\ &= \int_0^{50} \frac{2}{5} y \; dy + 1500 \text{ ft-lb} \\ &= 2000 \text{ ft-lb.} \end{aligned}$$