4.3 Families of Functions

We say that all functions of the form $y = a(x+b)^2 + c$ form a **family of functions**; their graphs are like that of $y = x^2$, except for shifts and stretches determined by the values of a, b and c. The constants a, b, c are called **parameters**. Different values of the parameters give different members of the family.

One reason for studying families of functions is their use in mathematical modeling. Confronted with the problem of modeling some phenomenon, a crucial first step involves recognizing families of functions which might fit the available data.

Motion Under Gravity

The position of an object moving vertically under the influence of gravity can be described by a function in the two parameter family

$$y = -4.9t^2 + v_0 t + y_0,$$

where t is the time in seconds and y is the distance in meters above the ground. Why are the parameters v_0 and y_0 important here? When will the object reach its maximum height?

The Bell-Shaped Curve

This family is related to the **normal density function**, used in probability and statistics. The family is given by

$$y = e^{-(x-a)^2/b},$$

where we assume that $b > 0$.

To see the role of the parameter a, fix $b = 1$, then the subfamily becomes

$$y = e^{-(x-a)^2}.$$

Thus, the role of a is to shift the graph of $y = e^{-x^2}$ to the right or left.

We now consider the role of the parameter b by studying the family with $a = 0$:

$$y = e^{-x^2/b}.$$

Let us investigate the critical points and the points of inflection of these curves. We calculate

$$\frac{dy}{dx} = -\frac{2x}{b} e^{-x^2/b},$$

and, using the product rule, we see

$$\frac{d^2y}{dx^2} = -\frac{2}{b} e^{-x^2/b} - \frac{2x}{b}\left(-\frac{2x}{b} e^{-x^2/b}\right) = \frac{2}{b}\left(\frac{2x^2}{b} - 1\right) e^{-x^2/b}.$$

The critical points occur where $dy/dx = 0$, i.e., where

$$\frac{dy}{dx} = -\frac{2x}{b} e^{-x^2/b} = 0.$$

Since $e^{-x^2/b} \neq 0$ for any x, the only critical point occurs at $x = 0$. At that point, $y = 1$ and $d^2y/dx^2 < 0$, so by the second derivative test, there is a local maximum at $x = 0$ (this is also a global maximum).

Inflection points occur where the second derivative changes sign; so we first find the x-values for which $d^2y/dx^2 = 0$. Again, since $e^{-x^2/b} \neq 0$ for any x, the only possibilities are when

$$\frac{2x^2}{b} - 1 = 0,$$

i.e., when $x = \pm\sqrt{\frac{b}{2}}$. Looking at the expression for d^2y/dx^2, we see that d^2y/dx^2 is negative for $x = 0$, and positive as $x \to \pm\infty$. Therefore, the concavity changes at $x = -\sqrt{b/2}$ and $x = \sqrt{b/2}$, so we have inflection points here.

Returning to the two-parameter family, $y = e^{-(x-a)^2/b}$, we conclude there is a maximum at $x = a$, obtained by horizontally shifting the maximum $x = 0$ of $y = e^{-x^2/b}$ by a units. There are inflection points at $x = a \pm \sqrt{b/2}$, obtained by shifting the inflection points $x = \pm\sqrt{b/2}$ of $y = e^{-x^2/b}$ by a units.

With this information we can see the effect of the parameters. The parameter a determines the location of the center of the bell and the parameter b determines how narrow or wide the bell is. If b is small, then the inflection points are close to a and the bell is sharply peaked near n; if b is large, the inflection points are farther away from a and the bell is spread out.

Exponential Model with a Limit

This family, given by

$$y = a(1 - e^{-bx}),$$

represents functions whose quantities are increasing but leveling off. For example, a body dropped in a thick fluid speeds up initially, but its velocity levels off as it approaches a terminal velocity. Similarly, if a pollutant pouring into a lake builds up toward a saturation level, its concentration may be described this way. The family might also represent functions describing the temperature of an object in an oven.

Recall from algebra that the a in $y = a(1 - e^{-bx})$ vertically stretches or shrinks the graph of $y = 1 - e^{-bx}$. Physically, the value of a represents the terminal velocity of a falling body, or the saturation level of the pollutant in the lake.

Similarly, (from algebra) the b in $y = a(1 - e^{-bx})$ horizontally stretches or shrinks the graph of $y = a(1 - e^{-x})$. The parameter b determines how sharply the curve the curve rises and how quickly it gets close to the line $y = a$.

The Logistic Model

The **logistic** family is often used to model population growth when it is limited by the environment. The family is given by

$$y = \frac{L}{(1 + Ae^{-kt})}.$$

We assume that $L, A, k > 0$ and we look at the roles of each of the three parameters in turn.

We can again use algebra to determine the roles of both L and k, they are vertical and horizontal stretch/shrinks, respectively. Here we provide a more detailed explanation:

The values of y level off as $t \to \infty$ since $Ae^{-kt} \to 0$ as $t \to \infty$. Thus, as t increases, the values of y approach L. The line $y = L$ is a horizontal asymptote, called the **limiting value** or **carrying capacity**, and representing the maximum sustainable population. The parameter L stretches or shrinks the graph vertically.

With L and A fixed, we see that varying k affects the rate at which the function approaches the limiting value L. If k is small, the graph rises slowly; if k is large, the graph rises steeply.

The parameter A alters the point at which the curve cuts the y-axis–larger values of A move the y-intercept closer to the origin. At $t = 0$ we have $y = L/(1 + A)$, confirming that increasing A decreases the value of y at $t = 0$.